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SOME MATHEMATICAL METHODS AND TECHNIQUES IN ECONOMICS[1]

BY

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That mathematics plays a certain role in various phases of economic theory is, of course, quite well known. The number of mathematicians and economists unaware of the discipline known as "Mathematical Economics" is surely small. Unfortuntely, the number of mathematicians and economists who are aware of the aims, the methods and the successes (the failures are, as is always the case, only too well known) of this rapidly expanding field, is likewise small.

The majority of us are exposed in the very early stages of our training to simple applications of mathematics in economics. To cite an instance, the classical high-school course in algebra which does not use the price-demand relation to illustrate the ideas of an inverse variation, is probably non-existent. A good many mathematicians think that this is almost the sum total of pure mathematical depth and sophistication that the economist encounters.

On the other hand, since statistics is, in so many obvious ways, ideally suited for analyzing certain types of economic phenomena, one is often fooled into believing that this represents all of the applied mathematical ideas that can play a central position in an economic investigation.

The appearance, in recent years, of the book "The Theory of Games and Economic Behavior" by von Neumann and Morgenstern [40] has been instrumental in dispelling from the minds of many mathematicians and economists some of the false ideas about what mathematics entered, and possibly more important, what mathematics does not enter, in the problems arising in everyday mathematical economics.

The purpose of this paper is, in part, to give a presentation of some phases of pure mathematics that are in current use in the economic world. We do not claim or pretend that this paper is either exhaustive or definitive—we merely propose to touch on a few boundary points with the hope that some of the readers will feel an urge to dig deeper into the interior. This paper is being written by a mathematician primarily for a math-

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ematical reader. The topics we discuss are thus selected, not so much for their economic depth or significance as for their mathematical interest.

Historically, the calculus entered very quickly into the study of price-demand relations. The calculus was, in fact, essentially the only tool used, until fairly recently in all of mathematical economics. (For a history and description of both the methodology and problems considered up to the beginning of the 20th Century see the authoritative article by V. Pareto [43]).

To say that the calculus, as a fundamental weapon, has by now played out its role would be both exaggerated and misleading. However, its initial foothold has been weakened considerably, and in certain places, arguments previously employing calculus techniques have been reworked using more powerful and more modern methods. There are several fundamental reasons for this trend to get away from using the calculus. Firstly, in a desire for generality, the conditions of differentiability placed on functions, especially when not needed, are aesthetically unsatisfactory. Secondly, the existence of derivatives for functions arising from a study of complex and oftentimes highly discrete situations is by no means a natural assumption. Thirdly the appropriate calculus conditions imposed on the functions considered at times obscure the essential nature of the problem. Added to this are the huge successes achieved in other fields (physics, for one) by the introduction of more of the full mathematical power available today.

This accounts for the tendency, with certain economists, to introduce some "natural" tools in their domain. The surprising effect (to some people) has been a great simplification and reinterpretation of old results and a satisfying surge forward in new research.

Although it is clear that mathematical statistics and game theory play a vital part in economic theory, we shall not consider their applications here. How and why they are used can be found very easily, and in many places [3, 33, 38, 40, 41, 50].

As we have pointed out previously, the calculus has performed (and is still performing) a striking function in the discipline. We begin the paper proper with an example that arises in economics and is handled via the calculus. This example has its origin in the theory of consumer's behavior; the derivation we present is of the so-called Slutsky Equation (Slutsky [47]; for a recent treatment of it see Samuelson [45, p. 97–103]). The problem itself is of classical stature in the economics of consumption. Its solution is a striking example of how a serious economic result emerges from an elementary mathematical analysis.[2]

We suppose there are n commodities which are labeled $1, 2, \cdots, n$ and a given consumer. A commodity bundle is a vector x whose ith coordinate represents an amount of the ith commodity. We suppose that we are also given prices for each commodity. Let p be the price vector. Given a commodity bundle x it is assumed that the consumer derives a certain satisfaction $s(x)$ where s is a real-valued, twice differentiable strictly convex function. (Economists usually consider a more general class of function, but for the sake of simplicity we restrict ourselves to the convex case). For economic purposes s could be equally well replaced by a monotonic increasing transformation of itself. In order for our results to be economically meaningful, our description of the consumer's behavior should be independent of the particular transformation applied to s. Finally, we suppose the consumer has a given amount of money, μ. Subject to the budgetary

[2]The form of the proof given here is due to G. Debreu.

constraint

$$p'x = \mu \qquad (\text{' denotes transposes}) \tag{1}$$

the consumer tries to maximize his satisfaction $s(x)$. Out of this desire, there results the necessary condition

$$\frac{ds}{dx} = \sigma p, \qquad \sigma \text{ a scalar and} \tag{2}$$

where σ, of course, depends on the particular form of s. (σ has traditionally been called the *marginal utility of money*).

On differentiating (2) we obtain

$$S\,dx = \sigma\,dp + p\,d\sigma \tag{3}$$

where S is the Hessian matrix of s. S is thus a symmetric matrix.

Differentiation of (1) yields

$$p'\,dx + (dp')x = d\mu. \tag{4}$$

Let

$$\Sigma = \begin{bmatrix} \frac{1}{\sigma} S & p \\ p' & 0 \end{bmatrix}.$$

From the symmetry of S, we have that Σ is also symmetric. The system (3) and (4) now becomes

$$\Sigma \begin{bmatrix} dx \\ -\frac{d\sigma}{\sigma} \end{bmatrix} = \begin{bmatrix} dp \\ d\mu - x'\,dp \end{bmatrix}. \tag{5}$$

Since s is strictly convex, Σ^{-1} exists. Since Σ is symmetric,

$$\Sigma^{-1} = \begin{bmatrix} X & \gamma \\ \gamma' & c \end{bmatrix},$$

where X is again symmetric and where γ is a column vector and c is a number. Equation (5) then gives rise to

$$\begin{bmatrix} dx \\ -\frac{d\sigma}{\sigma} \end{bmatrix} = \begin{bmatrix} X & \gamma \\ \gamma' & c \end{bmatrix} \begin{bmatrix} dp \\ d\mu - x'\,dp \end{bmatrix}. \tag{6}$$

Expanding the right-hand side of (6), we arrive at $dx = X\,dp + \gamma\,(d\mu - x'\,dp)$. Thus

$$\frac{\partial x}{\partial p} = X - \gamma x', \qquad \frac{\partial x}{\partial \mu} = \gamma.$$

Combining these we have

$$\frac{\partial x}{\partial p} = X - \frac{\partial x}{\partial \mu} x', \tag{7}$$

and so

$$X = \frac{\partial x}{\partial p} + \frac{\partial x}{\partial \mu} x'. \tag{8}$$

Since X is symmetric, picking out the (i, j) and (j, i) element we have

$$\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial \mu} = \frac{\partial x_j}{\partial p_i} + x_i \frac{\partial x_j}{\partial \mu}. \tag{9}$$

This is the Slutsky Equation. Notice that the result is independent of the particular transformation applied to s. The respective sides of (9) are termed the *substitution coefficients*. We now seek an economic interpretation for these. We ask: what changes in amounts of commodities and what changes in income would leave the consumer's satisfaction unchanged. From (2), since s is being kept constant, $p' \, dx = 0$. So (4) becomes $d\mu = x' \, dp$. Thus $dx = X \, dp$, whence the matrix elements of X, that is, the substitution coefficients describe the consumer behavior when satisfaction is assumed to be held constant.

This simple illustration of the use of the calculus, and of a calculus of the most rudimentary form, accounts, to a small degree, for the security the economist felt in the continued use of the calculus as his primary tool. As a consequence, he felt no particular urgency to broaden the mathematical repertoire to be applied to his problems. I should not like to imply that the preceding example is typical, in its mathematical depth, of the category of problems attacked. Mathematical analysis of a far more penetrating character was employed by some economic writers. Nontrivial existence theorems in the theory of differential equations, results from the theory of integral equations, and a great wealth of related mathematics was put to very effective use. (For an excellent treatment, along such lines, of a wide variety of topics, see Samuelson's stimulating book [45].)

However, in the late 1930's, the variety of mathematical topics finding application became more and more diversified and modern mathematical developments began reflecting on the nature of the work undertaken by the economist. In the remainder of this paper we shall touch on some of these. The order of their presentation is not meant to be chronological. These often involve convex sets or the calculus; this is quite natural, since such an important part of economics can be viewed as a maximizing activity.

We now consider a situation arising in the phase of economics known as "Welfare Economics." Before discussing the problem *per se* we need a brief, crude description of the general scope of the Welfare Economics. (For a much more thorough description see Samuelson [45], Arrow [1]).

Suppose that there are n-individuals in a community, designated by $1, 2, \cdots, n$. We suppose that each of these has a preference ordering by which he ranks the possible prospects or social states that can confront him; that is, to the ith individual there belongs a binary relation R_i defined on the set of possible social states X, Y, Z with the following properties:

1) XR_iY or YR_iX,

2) XR_iY, YR_iZ implies XR_iZ.

Here, R_i is a complete (or as it is sometimes called, a weak simple) ordering; XR_iY can be thought of as "X is at least as good as Y as far as the ith individual is concerned." Let XP_iY denote that XR_iY but not YR_iX. A social state X will be said to be optimal if the following is true: if for any i there exists a Y with YP_iX, then for some other individual j, XP_jY. That is, X is optimal if whenever a social state Y is preferred to X by some individual, then Y is not "at least as good as X" according to all other individuals. Another way of looking at this type of optimality can be formulated as: let XRY mean XR_iY for all $i = 1, 2, \cdots, n$; R then defines a partial order in the set of social states; a state X is now optimal if it is a maximal element in this partial order. The problem of the Welfare Economics is to give a description of (and a prescription sa to how to attain) such an optimal social state.

Here is a situation which has been handled within the last ten years by two totally different approaches—one in the framework of the orthodox calculus and the other in that of the theory of convex sets. In order to exhibit the differences (and the metamorphosis) in these two polar attacks on the problem we present two solutions of the same problem arising in the Welfare Economics. The first treatment we give is the calculus discussion, and is due to Lange [35].

We assume each individual's preference ordering R_i can be represented by a real valued function $u^{(i)}$, the ith person's utility function. The purpose is to maximize $u^{(i)}$, for each i, subject to the conditions that $u^{(j)}$ is kept constant for $j \neq i$.

We suppose the utility function of each individual to be a function of the amounts of each commodity that he gets; that is, if $x_j^{(i)}$ is the amount of the jth commodity held by the ith individual, $j = 1, 2, \cdots, m$, then $u^{(i)} = u^{(i)}(x_1^{(i)}, x_2^{(i)} \cdots, x_m^{(i)})$. We further assume that each $u^{(i)}$ is appropriately differentiable. Let $X_r = \sum_{i=1}^n x_r^{(i)}$ be the total amount of the rth commodity which is obtained by the whole community. The X_r are interrelated by a "technological function" $F(X_1, X_2, \cdots, X_m) = 0$. The problem then becomes: maximize $u^{(i)}(x_1^{(i)}, \cdots, x_m^{(i)})$ $i = 1, 2, \cdots, n$ subject to the constraints

$$1) \qquad u^{(j)}(x_1^{(j)}, \cdots, x_m^{(j)}) = \text{constant for } j \neq i,$$

$$2) \qquad X_r = \sum_{i=1}^{n} x_r^{(i)} \text{ for each } r = 1, 2, \cdots, m,$$

$$3) \qquad F(X_1, X_2, \cdots, X_m) = 0.$$

From the theory of Lagrange multipliers, this is equivalent to "extremizing" the unconstrained expression

$$\sum_{i=1}^{n} \lambda_i u^{(i)} + \sum_{r=1}^{m} \mu_r \left(\sum_{i=1}^{n} x_r^{(i)} - X_r \right) + \mu F(X_1, \cdots, X_r),$$

where the λ's and μ's are Lagrange multipliers. Differentiation and elimination yields

$$\frac{\partial u^{(i)}}{\partial x_r^{(i)}} \Big/ \frac{\partial u^{(i)}}{\partial x_s^{(i)}} = \frac{\partial F}{\partial X_r} \Big/ \frac{\partial F}{\partial X_s} \qquad \text{for all } i, r \text{ and } s.$$

Equivalently,

$$\frac{\partial u^{(i)}}{\partial x_r^{(i)}} \Big/ \frac{\partial u^{(j)}}{\partial x_r^{(j)}} = \frac{\partial u^{(i)}}{\partial x_s^{(i)}} \Big/ \frac{\partial u^{(j)}}{\partial x_s^{(j)}} \qquad \text{for all } i, j, r \text{ and } s.$$

Economically, $\partial u^{(i)}/\partial x_r^{(i)}$ is the "marginal utility of the rth commodity for the ith individual," and the above gives, as a necessary condition, that certain ratios of marginal utilities be equal. The result is invariant with regard to monotone increasing transformations of the utility functions. The Lagrange multipliers could be interpreted as prices, and the solution of the problem could be stated in terms of the existence of prices with certain properties. This will be done in the second treatment we give of this same problem using the theory of convex sets.

Before turning to the second version of this problem, we digress into several pathways suggested by factors which have arisen in the examples which we have already discussed.

In the situations which we have considered, the following kind of assumptions have been made: a complete ordering, occurring naturally as a preference ranking, confronts the economist; he, in turn, wishes to discuss the optimal behavior under this ordering. In order to do so he resorts to a real-valued function, having desirable differentiability properties. However, the final description of the optimal behavior is given in a form independent, to a major degree, of this function. Two questions immediately present themselves. Firstly, what conditions on the ordering insure the existence of such order-preserving functions? Secondly, since the final form of the solutions does not depend on these functions, can all these problems be handled without the artifice of a real-valued representation of the ordering?

The first of these problems, important as it seems to be, has received very little attention from the economist. In fact, some economists have even gone as far as to tactily assume that every complete ordering can be so represented. (The lexicographic ordering of the unit square offers an easy counter-example to this). In case the completely ordered set is, in addition, endowed with a "probability-mixing" operation, this problem has been thoroughly thrashed out by von Neumann-Morgenstern, Marschak [37], and Herstein-Milnor [25]. For the case in which no such mixture operation exists, H. Wold [54] did give conditions on the ordering which guaranteed the existence of an order-preserving function; however, his conditions were somewhat restrictive. In the mathematical literature, in a paper by Eilenberg [15] there is given a fairly general set of conditions for the existence of a continuous order-preserving function. This has recently been rederived by Debreu [10]. The result can be stated as follows:

Let S be a completely ordered topological space such that

I) S is separable and connected

II) for every $a_0 \in S$ the sets $\{a \in S \mid a \leqq a_0\}$, $\{a \in S \mid a_0 \leqq a\}$ are closed. Then there exists on S a continuous, real, order-preserving function.

Since the economist usually works in a finite-dimensional Euclidian space, or at worst, in a separable Hilbert space, Debreu's condition that his space be separable and connected is really not very restrictive. Incidentally, Debreu has recently shown that if (I) is replaced by: $(I)'$ S is perfectly separable, then the final result still holds.

As for the second problem, namely that of entirely avoiding the real-valued representation of the ordering, the alternative approach, which we are about to present, to the problem in Welfare Economics furnishes a typical method which has been successfully employed. Although this was hardly the main goal of the paper to be cited, it was an interesting by-product of the approach. The discussion which follows stems from a paper by Debreu [11].

Let us again suppose that there are m commodities labeled 1, 2, $\cdots$, m and n consumption units, labeled 1, 2, $\cdots$, n. The activity of a consumption unit is characterized by a vector x, the consumption vector, in the m-dimensional Euclidian space; the ith component of this vector being the amount of ith commodity used by the consumption unit. We suppose that each consumer has his individual complete order R_i defined on the set of all consumption vectors. It may happen that xR_iy and yR_ix without $y = x$; but if we define xI_iy if this does occur, this yields a proper equivalence relation, and the equivalence classes, $S_i(x)$, now form a linearly ordered set T_i. The satisfaction space $\mathcal{S}$ is introduced as the set of all vectors $S = (S_1, S_2, \cdots, S_n)$ where $S_i \varepsilon T_i$. In $\mathcal{S}$ a partial order is defined by

$$S^{(2)} = (S_1^{(2)}, S_2^{(2)}, \cdots, S_n^{(2)}) \geqq S^{(1)} = (S_1^{(1)}, S_2^{(1)}, \cdots, S_n^{(1)})$$

if and only if $S_i^{(2)}R_iS_i^{(1)}$ for every $i = 1, 2, \cdots, n$.

The production activity of the system is represented by an input vector $y = (y_1, y_2, \cdots, y_m)$, where y_i is the input (positive or negative) of the ith commodity. Technological conditions restrain y to belong to a set η. Let $\eta^{\min}$ be the set of efficient production vectors. A family of sets η_j, $j = 1, 2, \cdots, r$ is a decomposition of η if $\eta = \sum \eta_j$, the sum being in the sense of vector sums of sets. η_j characterizes activities of the jth production unit. If $X_i = X_i(s_i^0)$ denote the $\{x_i \mid s_i(x_i) \geqq s_i^0\}$ the ith consumption unit, $X = \sum X_i$ is the set of all total consumption vectors. $z = x + y, x \varepsilon X, y \varepsilon \eta$ is the total net consumption of the whole economy. Let z^0 be the vector whose components are the available amounts of each commodity. The economic system is constrained by

$$y \varepsilon \eta, \qquad z \leqq z^0 \qquad \text{in the vector sense.}$$

The goal of the economic system is to find an $S \varepsilon \mathcal{S}$, maximal in the partial order defined on $\mathcal{S}$, which is consistent with the above constraints. We now assume all the sets X_i, η_j are *convex* and *closed.*

$$Z = \sum X_i + \sum \eta_j \qquad \text{is then convex.}$$

Using the separation theorem for convex sets, namely, if two closed convex sets with interior points have only one point in common then there is at least one plane through that point separating the convex sets, one readily obtains the following result:

A necessary and sufficient condition for S^0 to be maximal (or for $z_0 = \sum x_i^0 + \sum y_j^0$ to be minimal) is the existence of a price vector $p > 0$ and a set of numbers a_i, $i = 1, 2, \cdots, n$ so that

α) x_i being constrained by $p'x_i \geqq a_i$, $s_i(x_i)$ reaches it maximum at x_i^0 for every i

β) y_j being constrained by $y_j \varepsilon \eta_j$, $p'y_j$ reaches its minimum at y_j^0 for every j.

Here p is a positive, normal vector to the separating plane. The theorem restates the following rules of behavior for consumption and production units: each consumption unit, subject to a budgetary constraint maximizes its satisfaction and each production unit, subject to the technological constraints, maximizes its profit.

We leave these problems concerning convexity, maximization under constraints and related topics and turn to a phase of economics which employes completely different mathematical techniques.

Matrix theory, to some degree, enters into many portions of many diverse fields.

Matrices are used in these as a pure notational device, as a compact and transparent representation of systems of linear equations, and in many other subsidiary, convenient roles, without use being made of essential matrix theory. In economics, also, matrices make their appearance in this costume. But deep, significant, intrinsic results of the matrix theory do play an important part. In the recent literature one finds a number of research papers in economics which employ certain corners of matrix theory in the fullest. (To cite a few examples, we refer the reader to Chipman [9], Metzler [39], Goodwin [23], Simon and Hawkins [24] and Solow [48].)

The mathematical origin of a good many of these results is a sequence of papers by the German mathematician Frobenius [19, 20]. These results have been rederived and somewhat extended in a simple, modern fashion by Wielandt [53] and Debreu and Herstein [13]. In these above-mentioned papers, among other results, are obtained theorems concerning the nature of the characteristic roots of a non-negative matrix A and the properties of $(sI - A)^{-1}$ for s a sufficiently large real number.

We now consider an economic situation in which these theorems function effectively. We follow here the treatment of a problem in the theory of international trade as given by Solow [48].

Suppose we consider n countries carrying the labels $1, 2, \cdots, n$. Let a_{ij} denote the marginal propensity (that is, the increase in imports from country i by country j per unit increase in income of country j) of the jth country to import from the ith country, and a_{ii} the marginal propensity to consume domestic goods. Furthermore, let x_i represent the national income of the ith country, c_i the autonomous expenditure in country i. Then we have, assuming linearity (which can be viewed as a first approximation for more general cases)

$$\text{i)} \qquad x_i = \sum_{j=1}^{n} a_{ij}x_j + c_i\ .$$

In matrix form this becomes

$$\text{ii)} \qquad (I - a)x = c,$$

where $a = (a_{ij})$; x, c are column vectors. Economic meaningfulness demands that the quantities x_i, c_i be all nonnegative. We assume that all $a_{ij} \geqq 0$.

Thus we are immediately forced to consider conditions under which the systems $x = (I - a)^{-1} c$ is solvable in nonnegative terms.

The linear equation system (ii) is easily seen to be the static solution of the linear difference equation system

$$\text{iii)} \qquad Ix(t + 1) - ax(t) = c.$$

A question of importance for this dynamic system will naturally concern the stability properties of its solutions. These can be shown to be equivalent to the condition that all the characteristic roots of a are less than 1 in absolute value. For this system Metzler [39] has proved: if $a_{ij} \geqq 0$ it is necessary and sufficient for the stability of (iii), that $I - a$ have all its principal minors positive. (As a pure matrix theorem this is an easy consequence of the results of Frobenius, when we use that stability is equivalent to the condition that characteristic roots of a be less than 1 in absolute value.) These types of questions are treated in Section 3 of the paper by Debreu and Herstein [13].

Let us consider $a = (a_{ij})$, a nonnegative, $n \times n$ matrix. A set, S, of indices will be said to be closed if $a_{pq} = 0$ for $q \in S$, $p \notin S$. That is, a closed set is associated with a collection of countries each of which spends in no country which is not in the collection. If no such proper closed set exists the matrix a is said to be indecomposable. This definition coincides with the purely mathematically motivated definition given by Frobenius [20], namely: the matrix a is said to be indecomposable if, for no permutation matrix π, the product $\pi a \pi^{-1}$ can be represented in the form

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where the A_{ii} are square submatrices. A matrix which is not indecomposable is called decomposable. If b is a decomposable matrix then there is a permutation matrix π so that

$$\pi b \pi^{-1} = \begin{bmatrix} B_1 & & * \\ & B_2 & \\ & & \ddots \\ 0 & & B_r \end{bmatrix}$$

where the B_i are square indecomposable submatrices on the diagonal.

In terms of the economic model, if the import matrix a is indecomposable, then a dollar spent in any one country will eventually induce spending in every other country.

Let us now assume that the import matrix a is indecomposable. From the results of Wielandt [53] or Solow [48] we can find a permutation π which puts a into the form

$$\pi a \pi^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & G_w \\ G_1 & 0 & \cdots & 0 & 0 \\ 0 & G_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & G_{w-1} & 0 \end{bmatrix},$$

where the 0's on the diagonal are square matrices. Economically this can be interpreted as: the countries in G_i spend only in those in G_{i+1}.

From the Frobenius theorems the following main result can be extracted: "if a is a nonnegative, indecomposable matrix then a has a positive characteristic root r so that

1) If α is any other characteristic root of a then $|\alpha| \leqq r$;
2) to r can be associated a positive characteristic vector;
3) r is a simple root;
4) an increase of any element of a yields an increase in r.

Moreover if there should be exactly k roots, α_λ, with $|\alpha_\lambda| = r$ then $\alpha_\lambda = r \exp(2\pi\lambda/k)$,

$\lambda = 1, 2, \cdots, k$ and a permutation π exists so that

$$\pi a \pi^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & A_k \\ A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{k-1} & 0 \end{bmatrix}.$$

Solow's w, which had a purely economic origin now can be interpreted as k, the number of characteristic roots of largest absolute value. After w time units have elapsed, the original expenditures from G_i in G_{i+1} has an influence in G_i, giving rise to a cyclicity of spending.

The above discussion encroaches on a tiny part of this territory, which has already been staked out by the economist. However, it is typical in the sense that the theorems of Frobenius play such a central role.

We now concern ourselves with several questions which have a combinatorial flavor to them. While these are not all phrased purely in economic terms, they have, or should have, many applications in economics.

The first of these is the so-called "personnel assignment" problem. Suppose that there are n individuals available to fill n positions. Suppose further that the ith individual obtains a rating of a_{ij} in his ability to handle the jth job. The question then is: how shall individuals be assigned to jobs in order to have the "overall efficiency" a maximum?

The problem can be put into purely mathematical garb in following vein: given a matrix A, for what permutation matrix P is the trace of AP a maximum? Of course this would merely require testing $n!$ possible permutations. Even for n relatively small this straightforward procedure would be out of the realm of practical feasibility.

The problem then becomes one of reducing the number of necessary computational steps. This can be achieved by (at least) two continuizations of the problem. We shall describe one of these, due to von Neumann [41] in very little detail, later in the paper. But before doing so, we should like to describe several related combinatorial questions.

A situation very similar to that in the personnel assignment problem is the "desk-cabinet" problem. This runs as follows: Suppose that there are n desks and n filing cabinets. Let d_{ij} be the distance from the ith desk to the jth cabinet. On the assumption that the individuals assigned to each desk make the same number of trips per day to their respective filing cabinets, what assignment of desks to cabinets should be made to insure that the total distance walked is a minimum. Formulated in mathematical terms, the problem is that of finding a permutation matrix P_0 which minimizes the trace of DP where D is the matrix of distances.

A purely economic variant of this can be phrased as follows: minimize the cost of production of n plants at n locations if d_{ij} is the cost of production for the ith plant at the jth location.

A generalization of the two above-mentioned situations, arising fairly naturally in economics, involves finding a permutation P_0 which minimizes or maximizes the trace of DP where P is no longer left free to roam over the whole symmetric group but is

restricted to range only over a subset thereof. In an actual problem this might be realized, for instance, by the condition that in any reshuffling of locations, railway depots would be restricted to occupy space only along railroad lines.

A solution to this problem would immediately yield a solution to another one, the "travelling salesman" problem. The travelling salesman problem, verbally described is the following: suppose a travelling salesman on his route must visit n cities and return to his home base which is one of these cities. If he must visit each of these cities once, and if he knows the distances between all of them, how should he plan his trip so that the distance he travels is minimal. This can be shown to be equivalent to minimizing the trace of CP where P is restricted to be a permutation which in its cycle decompostion is representable as an n cycle.

This relative of the Hamiltonian game can also be exhibited as a special case of a purely economic consideration due to Beckmann. Let there be n plants, and n fixed locations where these plants could be situated. Suppose a_{ij} represents the flow from the ith plant to the jth plant. Suppose also that k_{ij} is the cost of transportation from location i to location j. Then how should the plants be located in order that the total transportation costs among plants are a minimum. If $A = (a_{ij})$ and $K = (k_{ij})$ and if A' denotes the transpose, this simply becomes a question of finding a permutation P_0 which minimizes the trace of $A'P'KP$.

We return to the personnel assignment problem, and present a sketch of von Neumann's game theory approach to it [41].

Consider the following two person game: we have an $n \times n$ checker board, with each square having two indices, its row index and its column index. The first player picks a square; the second player then guesses either of the indices of the square which the first player has picked. He must state which index he is guessing. If he guesses correctly he receives an amount α_{ij} , where i, j are the indices of the square involved, from the first player. Otherwise he receives 0.

This game is related to the solution of the personnel assignment problem via the following theorem proved by von Neumann, (where the strategies refer to those of the first player).

"The extreme optimal strategies of the above game are precisely the following ones: Consider those permutations P_0 which maximize the trace of AP, where $A = (1/\alpha_{ij})$. For each P_0 assign the probability x_{ij} to the square i, j where $x_{ij} = (a/\alpha_{ij})\delta_{P_0(i),j}$ where δ is the Kronecker delta; and where a is the value of the game for the second player.

Using techniques of Brown and von Neumann [7] to get approximate solutions of games, the number of steps in solving the problem is reduced from $n!$ to a power of n.

Let $z = (z_{ij})$ be a vector in the n^2 Euclidian space. We define:

$$R = \{z = (z_{ij}) \mid z_{ij} \geqq 0, \qquad \sum_j z_{ij} = 1, \qquad \sum_i z_{ij} = 1\},$$

$$S = \{z = (z_{ij}) \mid z_{ij} \geqq 0, \qquad \sum_j z_{ij} \leqq 1, \qquad \sum_i z_{ij} \leqq 1\},$$

$$P = \{z = (z_{ij}) \mid z_{kj} = \delta_{P(i)}, \qquad j \text{ for some permutation } P\}.$$

Von Neumann's proof then hinges on the following two lemmas,

$$1) \qquad \{S = \mid z \qquad z \leqq w \text{ for some } w \in R\},$$

where $z \leqq w$ means the inequality is true in each component;

$$2) \qquad R = \text{convex hull of } P.$$

The reader will notice that certain very important current research in economics have scarcely been mentioned. For instance nothing has been said about the input-output models of Leontief or the applications of linear programming to economics. We felt that to do justice to these two would require much more space than would be appropriate for an article such as this. The reader is referred to Leontief's book, *The Structure of the American Economy*, [36] for the first topic, the book, *Application of Linear Programming to the Theory of the Firm*, by Dorfman, [14] for the second topic, and the book, *Activity Analysis of Production and Allocation* [29] for both.

With this discussion of the combinatorial problems we conclude the phase of the paper of "detailed" discussion of the applications of mathematics to economics. However, before concluding we wish to point out briefly some sources where one can find fine applications of other mathematical techniques.

The theories of differential equations and of difference equations play fundamental parts, in many connections, in economic theories. Chapter X of Samuelson's book [45] gives splendid illustrations of the use of techniques from these regions of mathematics.

Beckmann [5] has employed the classical theory of the calculus of variations in his study of continuous models of transportation. Earlier applications of the calculus of variation occur in papers by Hotelling [27] and Roos [44] and Evans [17].

Stone has used the theory of graphs in his economic studies. An example of this is [49]. The theory of graphs has also entered into the considerations of Koopmans and Reiter [31] of their transportation model. Charnes [8] has utilized the graph theory to extend certain computational techniques.

Following the trend towards axiomatization in mathematics there have been some purely axiomatic studies of economic questions; a pioneer effort in this direction is a postulational study of utility by Frisch [18]. Another example is the study of the existence of a social welfare function by Arrow [1] and Hildreth [26].

Many other applications of the theory of convex sets can be found than those already described in this paper. The theory of convex polyhedral cones was first developed by Weyl [52], and detailed mathematical investigations of these cones were carried out by Gerstenhaber [22]. These results, and many others, have a variety of applications, as the reader can see by looking at Koopmans [30], Georgescu-Roegen [21], Samuelson [46], and Arrow [2] in the monograph *Activity Analysis of Production and Allocation*. [29].

It is well known how the mathematician's interest in physical and astronomical problems have led to advances in mathematics itself. We cite here an example of a similar advance in mathematics which had as its origin a pure economic motivatiop. Not surprisingly, these new mathematical advances brought about have themselves stimulated the economic problem from which they sprang.

The Menger seminar in Vienna on mathematical economics, amongst many other topics, concerned itself with that of the existence of equilibrium. This led to the paper by A. Wald [51]. Soon after von Neumann [42] in proving the existence of an equilibrium point for an economic system, found it necessary to extend the Brouwer fixed point theorem. Kakutani [28] then simplified von Neumann's proof and cast the theorem in a somewhat different light. This lead to an even more powerful topological fixed point theorem by Eilenberg and Montgomery [16]. Begle [6] took up from there and gave a

very general fixed point theorem which subsumed that of Eilenberg and Montgomery. To complete the cycle, Arrow and Debreu [4, 12] have applied those theorems to obtain rigorous proofs of the existence of equilibrium points for fairly general economic systems.

In conclusion, the author should like to offer an apology to those economists whose contributions to the field may have been overlooked in this paper. The limitations of space, as well as the author's incomplete knowledge of the field, make such omissions unavoidable. An expository paper of this type cannot aim at complete coverage; rather, the author tried to convey an idea of the range and variety of mathematical results which have found applications in economics.

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AN ITERATIVE SOLUTION TO THE EFFECTS OF CONCENTRATED LOADS APPLIED TO LONG RECTANGULAR BEAMS*

BY

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Summary. The analysis of the bending of thin deep rectangular beams by concentrated loads has already been treated by L.N.G. Filon and Th. v. Kármán. These authors consider an infinitely long beam and express the load as a Fourier integral, obtaining integral solutions for the stresses and displacements. In their integral form, these results are rather difficult to interpret, although F. Seewald, using Kármán's analysis, has calculated numerical values for the case of a single concentrated load.

In this paper, a totally different approach to the problem is made. The loaded area is conformally transformed to a circle, and, using Muschelisvili's transformation of the boundary conditions, the problem is solved in the plane of the circle. The solution is then obtained in the form of a complex power series. In this manner, a direct solution, to any required degree of accuracy, is readily obtained.

1. Theory. Consider a long deep rectangular beam resting on two supports placed at the same level, and loaded by a weight placed midway between them. The weight of the beam will be neglected, and the beam will be assumed to be in a state of generalised plane stress, the plane of the mean stress being that of the length and depth. To determine the local effects of the concentrated loads on the stresses and deflexion it is convenient to move the supports to infinity and treat the beam as an infinitely long strip of depth $2h$ as is shown in Fig. 1. This procedure is that adopted by L.N.G. Filon, H. Lamb,

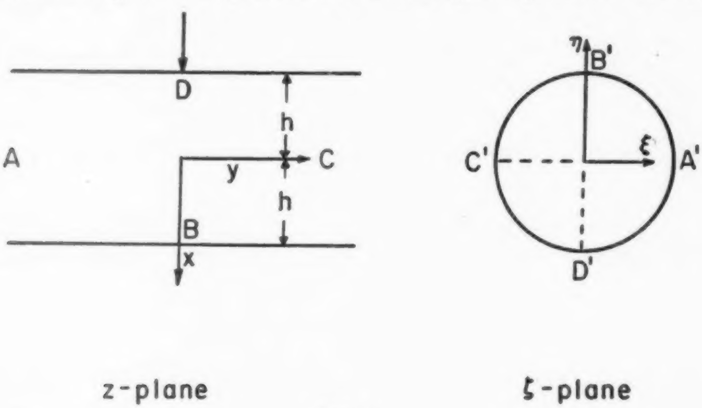

FIG. 1.

and Th. v. Kármán [1, 2] in their solutions of the problem.

Transform the strip ABCD to the unit circle, γ, in the ζ plane by the transformation

$$z = \frac{-2ih}{\pi} \log \frac{1 + \zeta}{1 - \zeta} = -4ih\pi^{-1}(\zeta + \zeta^3/3 + \zeta^5/5 + \cdots). \tag{1}$$

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Following Muschelisvili [3], define two functions $\Omega\,(z)$ and $w'(z)$ of the complex variable z by,

$$\sigma_x + \sigma_y = R\Omega'(z), \qquad \sigma_x - \sigma_y + 2i\tau_{xy} = -\tfrac{1}{2}\{z\bar{\Omega}''(\bar{z}) + \bar{w}''(\bar{z})\}, \tag{2}$$

where the dashes denote differentiation with regard to z and the bars denote the conjugate function. Then, the boundary conditions in the z plane can be expressed in the ζ plane by

$$\Omega[f(\sigma)] + f(\sigma)\bar{\Omega}_z'(\bar{\sigma}) + \bar{w}'[f(\bar{\sigma})] = F_1 + iF_2\,, \tag{3}$$

where,

$$z = f(\zeta) = \sum_{r=0}^{\alpha} u_r \zeta^r,$$

is the transformation relation, σ is the value of ζ on the unit circle,

$$\Omega_z'(\zeta) = \frac{d}{dz}\,\{\Omega[f(\zeta)]\},$$

$F_1 + iF_2$ is the transform of $f_1 + if_2$, $f_1 + if_2 = 4i\int_0^s (X_\nu + iY_\nu)$ ds, X_ν , Y_ν are the components of the boundary tractions, applied in the z plane, and s is the distance measured along the boundary in the z plane.

Since $\Omega_z'(\zeta)$ is analytic within the unit circle, we can express it by the Taylor's Series,

$$\Omega_z'(\zeta) = \sum_{r=0}^{r=\alpha} b_r \zeta^r. \tag{4}$$

Let us multiply both sides of (3) by $(1/2\,\pi i)\;d\sigma/(\sigma - \zeta)$ and integrate around γ. Using Cauchy's Theorem and, neglecting the constant, we have,

$$\Omega[f(\zeta)] = (1/2\pi i)\int_\gamma (F_1 + iF_2)\;d\sigma/(\sigma - \zeta) - (1/2\pi i)\int_\gamma f(\sigma)\bar{\Omega}_z'(\bar{\sigma})\;d\sigma/(\sigma - \zeta) \tag{5}$$

Similarly from the relation conjugate to (3) we obtain,

$$w'[f(\zeta)] = (1/2\pi i)\int_\gamma (F_1 - iF_2)\;d\sigma/(\sigma - \zeta) - (1/2\pi i)\int_\gamma \bar{f}(\bar{\sigma})\,\Omega_z'(\sigma)\;d\sigma/(\sigma - \zeta). \tag{6}$$

Substituting (4) into (5), carrying out the integration and noting that

$$\frac{d}{d\zeta}\,\frac{1}{2\pi i}\int_\gamma (F_1 + iF_2)\,\frac{d\sigma}{\sigma - \zeta}$$

can be expressed as a power series, $\sum_{r=0} p_r\,\zeta^r$, we obtain the following infinite set of equations

$$\sum_{0}^{n=\nu} (1 + \nu)u_{\nu+1}b_{n-\nu} + (1 + n)\sum_{\nu=0}^{\infty} u_{1+n+\nu}\bar{b}_\nu = p_n\,. \tag{7}$$

Solving equation (7) gives Ω_z' . To complete the solution, we require $w'(z)$ which is established in exactly the same way, using (6). The formula for $w'(z)$ is then

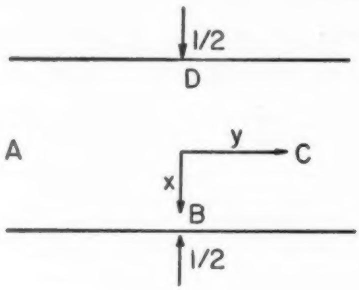
$$w'(z) = (1/2\pi i)\int_\gamma (F_1 - iF_2)\;d\sigma/(\sigma - \zeta) - \sum_{r=0}^{\infty} c_r \zeta^r, \tag{8}$$

where the c's are given by the relations,

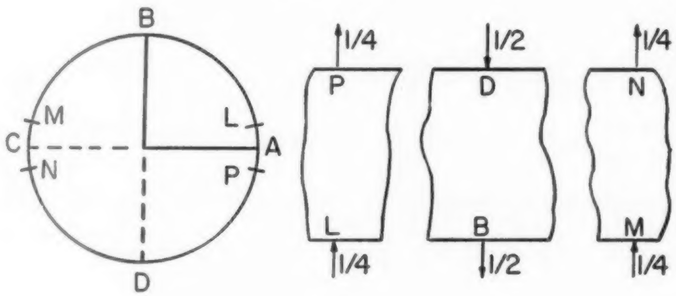
$$c_n = \sum_{r=0} \bar{u}_r b_{n+r}\,. \tag{9}$$

A full and complete account of the derivation of (7) and (8) is given in Reference 4.

2. Solution. Returning to the original problem, consider the given loading as a combination of the following two systems, where in system B, the points L,M,N,P are supposed to be very remote.



System A



System B

FIG. 2.

Case A. Load function:

$$\text{along } BCD,\ F_1 + iF_2 = -2i,$$
$$\text{along } DAB,\ F_1 + iF_2 = 0.$$

Then,

$$(1/2\pi i) \int_\gamma (F_1 + iF_2)\, d\sigma/(\sigma - \zeta) = -\pi^{-1} \log\, (\sigma_D - \zeta)/(\sigma_B - \zeta)$$

$$\frac{d}{d\zeta} (1/2\pi i) \int_\gamma (F_1 + iF_2)\, d\sigma/(\sigma - \zeta) = \frac{2i}{\pi} (1 + \zeta^2)^{-1}$$

$$(1/2\pi i) \int_\gamma (F_1 - iF_2)\, d\sigma/(\sigma - \zeta) = \pi^{-1} \log\, (e^{i3\pi/2} - \zeta)/(e^{i\pi/2} - \zeta).$$

Equations (7) become

$$2b_0 + .33333b_2 + .2b_4 + .142857b_6 + .11111b_8 + \cdots = -(2h)^{-1},$$

$$2b_0 + 1.6b_2 + .42857b_4 + .33333b_6 + .27273b_8 + \cdots = (2h)^{-1},$$

$$2b_0 + 1.7108b_2 + 1.5556b_4 + .45455b_6 + .38462b_8 + \cdots = -(2h)^{-1},$$

$$2b_0 + 1.778b_2 + 1.6364b_4 + 1.5385b_6 + .46666b_8 + \cdots = (2h)^{-1}, \text{ etc.}$$

Using an iterative process to solve these equations we obtain for the 6th and 7th iterations the following values:

Iteration	$b_0 \times h$	$b_2 \times h$	$b_4 \times h$	$b_6 \times h$	$b_8 \times h$	$b_{10} \times h$
6	−.32113	.91461	−1.0132	.98457	−.98724	.98556
7	−.34898	.88018	−1.0381	.97263	−1.0143	1.0143

From these we obtain

$$\Omega_z' = -.329/h + .895\zeta^2/h - 1.025\zeta^4/h + .979\zeta^6/h + \cdots,$$

$$= -\frac{1}{2h}\frac{1-\zeta^2}{1+\zeta^2} + .171/h - .105\zeta^2/h - .025\zeta^4/h + \cdots, \tag{a}$$

$$\Omega_z'' = \frac{d\Omega'}{d\zeta}\frac{d\zeta}{dz} = \frac{i\pi}{4h^2}\zeta(1-\zeta^2)\left[\frac{2}{(1+\zeta^2)^2} - .21 - .10\zeta^2 - .126\zeta^4 + \cdots\right]. \tag{b}$$

Using these values in equations (8) and (9) we calculate $w'(z)$ and $w''(z)$ as follows:

$$w'(z) = \frac{1}{\pi}\log\left(\frac{-i-\zeta}{i-\zeta}\right) - \frac{i\zeta}{2}\frac{1-\zeta^2}{1+\zeta^2} + \frac{2i}{\pi}[-.55014\zeta + .771667\zeta^3$$
$$+ .043\zeta^5 + \cdots], \tag{c}$$

$$w''(z) = \frac{1-\zeta^2}{h}\left[\frac{1.2854}{1+\zeta^2} - \frac{\pi}{2}\left[\frac{\zeta}{1+\zeta^2}\right]^2 - .11763 + .1075\zeta^2 + .107\zeta^4 + \cdots\right] \tag{d}$$

Case B. Load function:

For	*BM*	*MCN*	*ND*	*DP*	*PAL*	*LB*
$F_1 + iF_2$	0	$-i$	$-2i$	0	$-i$	$-2i$

Then,

$$\frac{1}{2\pi i}\int_\gamma (F_1 + iF_2)\frac{d\sigma}{\sigma - \zeta} = \frac{1}{2\pi i}\left[-i\log\left(\frac{\sigma_N-\zeta}{\sigma_M-\zeta}\right)\left(\frac{\sigma_D-\zeta}{\sigma_N-\zeta}\right)\left(\frac{\sigma_L-\zeta}{\sigma_P-\zeta}\right)\left(\frac{\sigma_P-\zeta}{\sigma_L-\zeta}\right)^2\right]$$

Now when L and P move to A $\sigma_L = \sigma_P = 1$, and when M and N move to C $\sigma_M = \sigma_N = -1$. Thus,

$$\frac{1}{2\pi i}\int_\gamma (F_1 + iF_2)\frac{d\sigma}{\sigma - \zeta} = \frac{-1}{\pi}\log\frac{-1-\zeta^2}{1-\zeta^2}.$$

The constants b are purely imaginary and equations (7) become

$$b_1 - 2b_1/3 - 2b_3/5 - 2b_5/7 - 2b_7/9 + \cdots = -1/h,$$
$$b_1 + b_3 - 4b_1/5 - 4b_3/7 - 4b_5/9 - 4b_7/11 + \cdots = 0,$$
$$b_1 + b_3 + b_5 - 6b_1/7 - 6b_3/9 - 6b_5/11 - 6b_7/13 + \cdots = -1/h, \text{ etc.}$$

Assume

$$\Omega_z' = i\pi b_1 z/4h + f_3\zeta^3 + f_5\zeta^5 + \cdots,$$

i.e., assume that the b_r are of the form

$$b_1 = b_1,$$
$$b_3 = b_1/3 + f_3,$$
$$b_5 = b_1/5 + f_5, \text{ etc.}$$

Then since

$$1 - \frac{2}{3} - \frac{2}{5.3} - \frac{2}{7.5} - \frac{2}{7.9} + \cdots = 0,$$
$$1 + \frac{1}{3} - \frac{4}{1.5} - \frac{4}{3.7} - \frac{4}{5.9} + \cdots = 0,$$
$$1 + \frac{1}{3} + \frac{1}{5} - \frac{6}{1.7} - \frac{6}{3.9} - \frac{6}{5.11} + \cdots = 0, \text{ etc.,}$$

b_1 can be eliminated from the equations and we are left with the following system for the values of f:

$$-.2f_3 - .142857f_5 - .111111f_7 - .090909f_9 + \cdots = -1/2h,$$
$$.107143f_3 - .111111f_5 - .090909f_7 - .076923f_9 + \cdots = 0,$$
$$.055556f_3 + .075758f_5 - .076923f_7 - .066667f_9 + \cdots = -1/6h, \text{ etc.}$$

To bring these into a form suitable for the iteration process subtract each equation from the one above it. Thus,

$$-.307143f_3 - .031746f_5 - .020202f_7 - .013986f_9 + \cdots = -1/2h,$$
$$.051587f_3 - .186869f_5 - .013986f_7 - .010256f_9 + \cdots = 1/6h,$$
$$.021365f_3 - .027681f_5 - .135256f_7 - .007843f_9 + \cdots = -1/6h, \text{ etc.}$$

From these we obtain

$f_3 \div i/h$	$1.581 \pm .004$	$f_9 \div i/h$	$-.72 \pm .01$
$f_5 \div i/h$	$-.581 \pm .001$	$f_{11} \div i/h$	$1.25 \pm .01$
$f_7 \div i/h$	$1.33 \pm .02$	$f_{13} \div i/h$	$-.77 \pm .02$

Thus

$$h\Omega_z' = ib_1 z\pi/4 - i\zeta/(1+\zeta^2) + i\zeta(1 + .581\zeta^2 + .419\zeta^4 + .33\zeta^6 + \cdots) \tag{e}$$

and

$$h^2\Omega_2'' = ib_1 h\pi/4 + \frac{\pi}{4}(1-\zeta^2)^2/(1+\zeta^2)^2 - \frac{\pi}{4}(1-\zeta^2)[1 + 1.743\zeta^2 + 2.095\zeta^4 + \cdots] \tag{f}$$

From formulae 8 and 9 and using the tabulated values of f, we have,

$$w'(z) = ib_1 z^2\pi/8h + \pi^{-1}\log(1+\zeta^2)/(1-\zeta^2) + i - (1+\zeta^2)^{-1} + (4/\pi)[c_0 + .972\zeta^2 + .749\zeta^4 + \cdots], \tag{g}$$

and

$$hw''(z) = i\zeta/(1+\zeta^2) + (i\pi/2)\zeta(1-\zeta^2)/(1+\zeta^2)^2 + ib_1 z\pi/4 + i\zeta(1-\zeta^2)(1.944 + 2.996\zeta^2 + \cdots). \tag{h}$$

The stress distribution for which $\Omega_z'' = ib_1 z\pi/4h$, $w''(z) = ib_1 z\pi/4h$ is that due to pure flexure, provided $b_1 = -iB_1/h$, where B_1 is real. From statics, we know that there is an infinite bending moment at the centre section due to the loads at infinity. But, from statics, at a section where the support loads are applied, the normal stress distribution must be equivalent to a moment of $h/2\pi$ as shown in Fig. 3. As this moment is applied

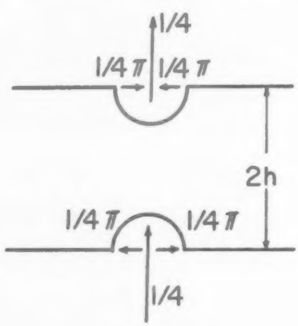


FIG. 3.

effectively at infinity, it must be included in the b_1 term. For as $\zeta \to \pm 1$, $R\Omega_2' \to B_1 z/4h^2$. Hence, B_1 is made up of two parts, that due to an infinite moment which will be represented by b' and that due to a moment of amount $h/2\pi$. Thus

$$B_1 = b' + 3/\pi^2$$

and

$$h\Omega_z' = b'z\pi/4h + 3z/4\pi h - i\zeta/(1+\zeta^2) + i\zeta(1 + .581\zeta^2 + .419\zeta^4 + \cdots), \tag{i}$$

$$w'(z) = b'z^2\pi/8h^2 + 3z^2/8\pi h^2 + \pi^{-1}\log(1+\zeta^2)/(1-\zeta^2) + i - (1+\zeta^2)^{-1} + (4/\pi)(c_0 + .972\zeta^2 + \cdots) \tag{j}$$

Equations a to f now give the solution to the problem. To calculate the stresses at any point we use equation (1) with the values of Ω_z', Ω_z'' and w'' obtained by substituting the corresponding values of ζ in the relevant equation a to j. Fig. 4 gives the distribution

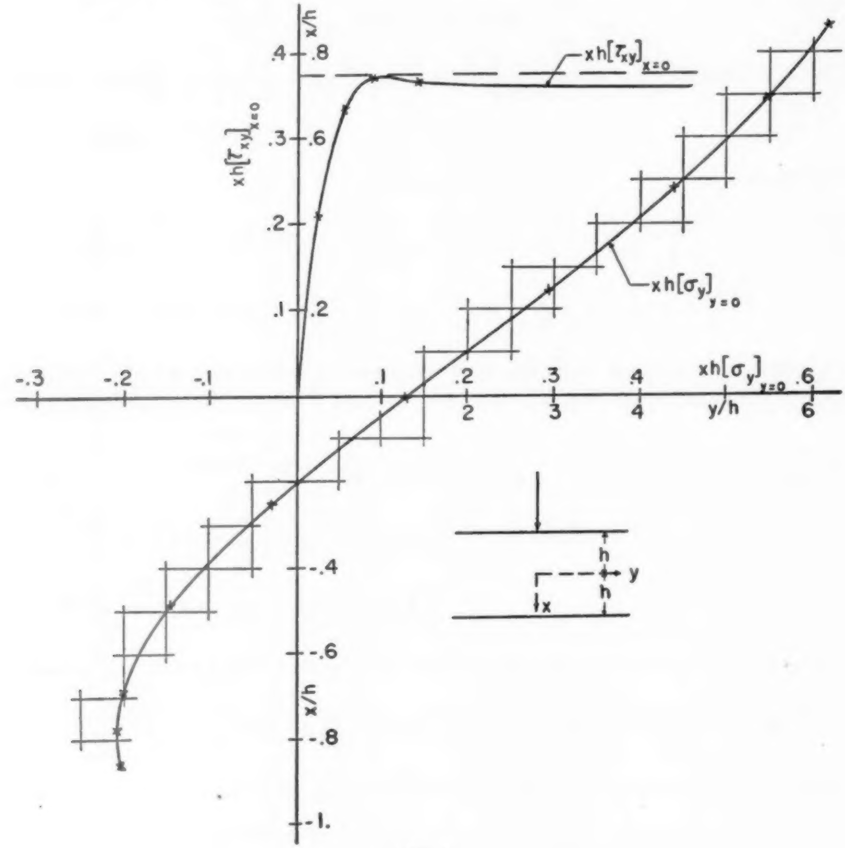


FIG. 4.

of normal stresses across BD and the shear stress distribution along the y axis.

The deflections u and v are readily obtained from formula (10)

$$8\mu(u + iv) = (3 - \nu)/(1 + \nu)\Omega(z) - z\overline{\Omega}'(\bar{z}) - \overline{w}'(\bar{z}), \tag{10}$$

where μ is the shear modulus; ν is Poisson's ratio, u, v are the components of displacement in the x and y direction respectively; and D is $u + iv$.

We are particularly concerned with the displacement and curvature of the y axis and for this case some simplification of 10 is possible.

Differentiate both sides of (10) with regard to y, then,

$$8\mu \frac{\partial D}{\partial y} = i\Omega_z'(3 - \nu)/(1 + \nu) - i\overline{\Omega}'(\bar{z}) + iz\overline{\Omega}''(\bar{z}) + i\overline{w}''(\bar{z}).$$

Since it is only the loading condition of case B that will have any effect on the centre line we have for $x = 0$,

$$\zeta = \xi,$$

$$i\Omega'(z) = -i\bar{\Omega}'(\bar{z}),$$

$$i[z\bar{\Omega}''(\bar{z}) + \bar{w}''(\bar{z})] = \frac{-\pi}{4h^2} \cdot y \cdot \left(\frac{1-\xi^2}{1+\xi^2}\right)^2 + \frac{1}{h} \cdot \frac{\xi}{1+\xi^2} + \frac{\pi}{2h} \cdot \frac{(1-\xi^2)}{(1+\xi^2)^2}\xi$$
$$+ .944\xi/h - .024\xi^2/h + \cdots.$$

and therefore

$$8\mu \frac{\partial u}{\partial y}\bigg]_{x=0} = \frac{4}{1+\nu}\left[\frac{-\pi b' y}{4h^2} + \frac{1}{h} \cdot \frac{\xi}{1+\xi^2} - (.696\xi + .480\xi^3 + \cdots)\right] - \frac{\pi}{4h^2} \cdot y$$
$$\cdot \left(\frac{1-\xi^2}{1+\xi^2}\right)^2 + \frac{1}{h} \cdot \frac{\xi}{1+\xi^2} + \frac{\pi}{2h} \cdot \xi \cdot \frac{1-\xi^2}{(1+\xi^2)^2} + (\xi/h)[.944 - .024\xi + \cdots].$$

To compare this result with the usual engineering form let u_1 be the deflection of $x = 0$ and, considering the case $y > 0$ let $\xi = -t$ and E denote Young's Modulus. Then,

$$\frac{du_1}{dy} = \frac{\pi b'}{4h} \cdot \frac{y}{Eh} + \frac{3}{8} \cdot \left(\frac{y}{h}\right)^2 \frac{1}{Eh} - \frac{1}{Eh} \cdot \frac{t}{1+t^2} + \frac{1}{Eh}[.696t - .608t^2$$
$$+ .480t^3 - .405t^4 + .358t^5 + \cdots] - \frac{1}{8\mu h}\left[\frac{\pi}{4} \cdot \frac{y}{h} \cdot \left(\frac{1-t^2}{1+t^2}\right)^2 + \frac{t}{1+t^2}\right.$$
$$\left. + \frac{\pi}{2} \cdot t \cdot \frac{1-t^2}{(1+t^2)^2} + .944t - .024t^2 + \cdots\right]$$

Using Euler's summation process we have for the first series on the right-hand side

$$.696t - .608t^2 + .480t^3 + \cdots = .696\frac{t}{1+t} + .088\left(\frac{t}{1+t}\right)^2 + \cdots.$$

Then if Δ is the correction to the usual engineering deflection,

$$\frac{d\Delta}{dy} = \frac{-1}{Eh} \cdot \frac{t}{1+t^2} + \frac{1}{Eh}\left(.696\frac{t}{1+t} + .088\left(\frac{t}{1+t}\right)^2 + \cdots\right)$$
$$- \frac{1}{8\mu h}\left[\frac{\pi}{4} \cdot \frac{y}{h}\left(\frac{1-t^2}{1+t^2}\right)^2 + \frac{t}{1+t^2} + \frac{\pi}{2} \cdot t \cdot \frac{1-t^2}{(1+t^2)^2} + .944t - .024t^2 + \cdots\right]$$

Hence the correction to the usual engineering curvature is

$$K = \frac{d^2\Delta}{dy^2} = \frac{\pi}{4h^2} \cdot \frac{1-t^2}{E}\left(.696 - 1.216\frac{t}{1+t} + .224\left(\frac{t}{1+t}\right)^2 + \cdots\right)$$
$$- \frac{\pi}{4h^2E}\left\{\frac{1-t^2}{1+t^2}\right\}^2 - \frac{\pi}{32\mu}\frac{1-t^2}{h^2} \cdot \left[.944 + 2\frac{1-t^2}{(1+t^2)^2} - \frac{\pi y}{4h} \cdot \frac{8t(1-t^2)}{(1+t^2)^3}\right.$$
$$\left. + \frac{\pi}{2}\frac{1-6t^2+t^4}{(1+t^2)^3}\right]$$

and,

$$\Delta = \frac{-1}{\pi E} \log \frac{1 + t^2}{1 - t^2} + \frac{1}{E}\left[.370 \frac{y}{h} + \frac{1.656}{\pi} \cdot \frac{1}{1 + t} - \frac{.088}{\pi} \frac{1}{1 + t^2} + \cdots\right]$$

$$- \frac{1}{8\mu} \frac{y}{h} \cdot \frac{t}{1 + t^2} + \frac{1}{8\mu} \cdot \frac{1}{1 + t^2} + \frac{.236}{\pi\mu} \log (1 - t^2) - \frac{1}{8\mu} - \frac{1.568}{\pi E}.$$

Figure 5 gives a plot of K with y assuming $\nu = 0.3$ and shows that for $y > h$, K is

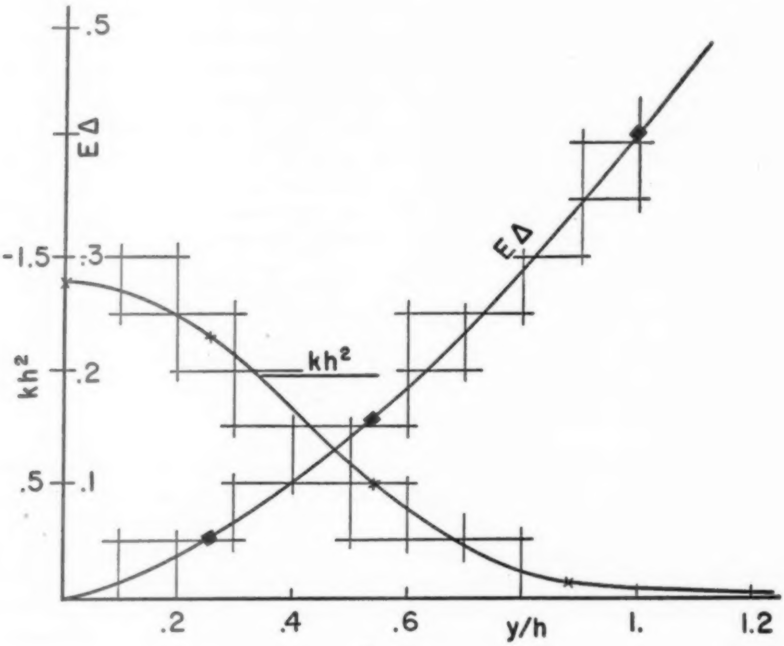


Fig. 5.

negligible. Hence the engineering approximation, that the curvature is proportional to the bending moment, is quite exact at a distance from the load greater than half the depth of the beam. Making the assumption that for $y > h$ the curvature of the centre line is proportional to the bending moment, we can then calculate the correction Δ which is also given in Fig. 5.

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MODIFIED STURM-LIOUVILLE SYSTEMS*

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1. Heat conduction problem. Consider the steady-state temperatures $U(x,z)$ of a thin slab of material in the shape of a square with edges at $x = 0$, $x = 1$, $z = 0$, and $z = 1$. Let the edge at $x = 0$ be kept at temperature zero and each point of the edge at $z = 0$ kept at temperature $F(x)$. Suppose the edge at $x = 1$ is in perfect thermal contact with a thin strip of anisotropic material the coefficient of conductivity of which is much greater in the x direction than in the z direction so that the temperatures in the strip can be regarded as a function of z alone. Suppose that an analogous situation holds for the edge at $z = 1$ so that the temperatures in that strip may be regarded as a function of x alone. If at the faces heat is transferred according to a linear law into an external medium at temperature zero and the coefficient of heat transfer $r(x)$ is a continuous function, the boundary value problem may be written**

$$
\begin{gathered}
U_{xx} + U_{zz} - cr(x)U = 0, \\
U(0, z) = 0, \\
U_x(1, z) - k_1 U_{zz}(1, z) = 0, \\
U(x, 0) = F(x), \\
U(x, 1) - k_2 U_{xx}(x, 1) = 0.
\end{gathered}
$$

In this statement of the problem $1/c$ is the conductivity of the slab, $k_1 = cq_1$, $k_2 = cq_2$, where q_1 and q_2 are the coefficients of conductivity of the strips along $x = 1$ and $z = 1$ in the z and x directions respectively.

The assumption made above of a narrow (relative to the size of the slab) strip of material whose temperature varies in only one direction may be closely realized in practice. For example, consider a material such as concrete which has imbedded in it pipes containing moving water. Also, consider a laminated material such as a brick wall.

In the solution of the problem we are led to the classical method of separation of variables, since, because of the boundary conditions at $x = 1$ and $z = 1$ and the non-constant coefficient in the differential equation, there is little hope for success in the employment of a finite Fourier transform or the Laplace transform. In deriving solutions of the homogeneous conditions of the form $U(x,z) = y(x)w(z)$ we obtain the following eigenvalue problem in $y(x)$:

$$
\begin{gathered}
y''(x) + [\lambda - cr(x)]y(x) = 0, \\
y(0) = 0, \qquad (2) \\
ck_1 r(1)y(1) - y'(1) - k_1 y''(1) = 0.
\end{gathered}
$$

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**Throughout this paper function values at an end point of an interval shall mean the limit the function assumes as the point is approached from the interior of the interval.

In order to justify the classical method of satisfying the non-homogeneous boundary condition $U(x,0) = F(x)$ of problem (1) by combining solution of the type $y(x)w(z)$, an expansion theorem for the eigenvalue problem (2) is required. In this paper we shall deal with properties of and expansion theorems for problems of which type (2) is typical.

2. The modified Sturm-Liouville problem. The eigenvalue problem (2) is a special case of the problem consisting of the Sturm-Liouville equation

$$y''(x) + [\lambda + p(x)]y(x) = 0 \tag{3}$$

in which $p(x)$ is a real, continuous function for $0 \leqq x \leqq 1$, with the boundary conditions

$$\begin{aligned} U_0[y] &\equiv a_1y(0) + a_2y'(0) + a_3y''(0) = 0, \\ U_1[y] &\equiv b_1y(1) + b_2y'(1) + b_3y''(1) = 0 \end{aligned} \tag{4}$$

in which the a's and the b's are real constants. In view of the differential equation (3), the boundary conditions may be written in the alternative form

$$\begin{aligned} U_{\lambda 0}[y] &\equiv \{a_1 - a_3[\lambda + p(0)]\}y(0) + a_2y'(0) = 0, \\ U_{\lambda 1}[y] &\equiv \{b_1 - b_3[\lambda + p(1)]\}y(1) + b_2y'(1) = 0. \end{aligned} \tag{5}$$

In this form the boundary conditions are of the form of the usual Sturm-Liouville problem except for the appearance of the parameter λ in the coefficients.

For two eigenfunctions y_i and y_j corresponding to distinct eigenvalues λ_i and λ_j we obtain by a familiar procedure the Green's formula

$$(\lambda_i - \lambda_j) \int_0^1 y_iy_j\, dx = [y_iy_j' - y_jy_i']_0^1 .$$

Supposing that $a_2 \neq 0$ and $b_2 \neq 0$ we substitute from (5) into the right member and obtain the orthogonality relation

$$\int_0^1 y_iy_j\, dx - \frac{a_3}{a_2} y_i(0)y_j(0) + \frac{b_3}{b_2} y_i(1)y_j(1) = 0. \tag{6}$$

It is easily verified that if $a_2 = 0$ or $b_2 = 0$ the term of the orthogonality relation involving that quantity does not appear.

Expansion theorems for various cases of the problem (3)(4), or the equivalent problem (3)(5), have been established by Langer [7], Gaskell [4], and Churchill [1]. Boundary value problems giving rise to the eigenvalue problem are described in the first two of these papers. Among these three papers, only Churchill's deals with the problem* involving the differential equation with non-constant coefficients. In this paper we shall establish an expansion theorem for the problem having non-constant coefficients in the differential equation. The author believes that the theoretical development leading up to the expansion theorem as well as the theorem itself helps fill the gap in theory between eigenvalue problems which involve the parameter in the coefficients of the boundary conditions and those which do not.

*Another paper by Langer [8] presents, in another way, derivations of some properties of eigenvalue problems more general than ours.

3. Definitions and properties of normal and positive-definite eigenvalue problems. We make the following definitions:

Definition 1. A function $u(x)$ shall be called a "V-function" if it is real, not identically zero, of class C^2, and satisfies the boundary conditions (4).

Definition 2. The inner product corresponding to the eigenvalue problem (3), (4) for two bounded integrable functions $f(x)$ and $g(x)$ is defined as follows:

$$[f, g] = \int_0^1 fg\,dx - \frac{a_3}{a_2} f(0)g(0) + \frac{b_3}{b_2} f(1)g(1).$$

We note here that the inner product is defined for any two V-functions. Also, the orthogonality relation (6) is simply $[y_i, y_j] = 0$ and the Fourier coefficient with respect to the function $f(x)$ and corresponding to the eigenfunction $y_i(x)$ is written formally as

$$\frac{[f, y_i]}{[y_i, y_i]}.$$

The inner product has the important properties of linearity and commutativity with respect to its functional arguments.

Definition 3. The eigenvalue problem (3), (4) is "normal" if for every V-function $u(x)$ it is true that

$$[u, u] > 0.$$

It is easy to verify that a necessary and sufficient condition that the problem (3), (4) is normal is that $a_2a_3 \leqq 0$ and $b_2b_3 \geqq 0$.

Theorem 1. If the eigenvalue problem (3), (4) is normal, it has only real and simple eigenvalues and real eigenfunctions.

Proof. Suppose λ is a complex eigenvalue, y the corresponding eigenfunction, and the imaginary part of λ is not zero. Then $\bar{y}$, the conjugate of y, is an eigenfunction and, according to the orthogonality relation (6), we have $[y,\bar{y}] = 0$. But this is a contradiction because the eigenvalue problem is normal ($a_2a_3 \leqq 0$, $b_2b_3 \geqq 0$) and the integral term of the inner product is greater than zero while the last two terms are greater than or equal to zero. Therefore, if the problem (3), (4) is normal it has only real eigenvalues.

Suppose next that $u(x)$ and $v(x)$ are two eigenfunctions both corresponding to the same real eigenvalue. Since both of these functions are solutions of (3), the Wronskian of the two functions is a constant. If this constant is evaluated by means of either boundary condition in (5), it is found to be equal to zero. The functions u and v are, therefore, linearly dependent. That is, any eigenvalue of (3), (4) is simple in the sense that there cannot be two linearly independent eigenfunctions corresponding to it. It follows immediately that any complex eigenfunction $u(x) + iv(x)$ is equal to $(k + i)v(x)$ for some k; any eigenfunction is real up to a constant factor.

Let us now prove that the eigenvalues are simple in the sense that they are not double roots of the characteristic equation. Let $y(x,\lambda)$ be a solution of the differential equation (3) such that it satisfies the boundary conditions

$$U_{\lambda 0}[y(x, \lambda)] = 0, \; y(1, \lambda) = 1.$$

The function $y(x,\lambda)$ and its first and second derivatives are continuous functions of x and analytic functions of λ, for the system defining $y(x,\lambda)$ is an initial value problem

with a complex parameter λ, and the coefficients of the problem are analytic functions of λ. Therefore we may expand $y(x,\lambda)$ in the following Taylor series about the eigenvalue λ_i :

$$y(x, \lambda) = y_i(x) + \sum_{n=1}^{\infty} A_n(x, \lambda_i)(\lambda - \lambda_i)^n, \tag{7}$$

where $y_i(x) = y(x,\lambda_i)$ is the eigenfunction corresponding to λ_i . Similar expressions can be written for $y'(x,\lambda)$ and $y''(x,\lambda)$.

In what follows we shall need to know that the coefficient of $(\lambda - \lambda_i)$ in the expansion of $y'(x,\lambda)$ is the same as $A_1'(x,\lambda_i)$. This is true if

$$\frac{\partial}{\partial\lambda}\frac{\partial}{\partial x} y(x, \lambda) = \frac{\partial}{\partial x}\frac{\partial}{\partial\lambda} y(x, \lambda) \tag{8}$$

which, in turn, is true if $(\partial/\partial\lambda)(\partial/\partial x)y(x,\lambda)$ is a continuous function of x and λ. Since $y'(x,\lambda)$ is analytic in λ,

$$\frac{\partial}{\partial\lambda} y'(x, \lambda) = \frac{1}{2\pi i}\int_C \frac{y'(x, \lambda)}{(z - \lambda)^2}\, dz,$$

where C is a finite closed path around λ. This expression shows that because $y'(x,\lambda)$ is continuous in x and λ, $(z - \lambda)^2$ is bounded from zero, and C is finite, $(\partial/\partial\lambda)y'(x,\lambda)$ is continuous in x and λ and (8) is valid. In a similar way it can be shown that $A_1''(x,\lambda_i)$ is the same as the coefficient of $(\lambda - \lambda_i)$ in the expansion of $y''(x,\lambda)$.

The problem which defines $y(x,\lambda)$ can be written as follows:

$$\begin{gathered} y'' + (\lambda_i + p)y = -(\lambda - \lambda_i)y, \\ \{a_1 - a_3[\lambda_i + p(0)]\}y(0, \lambda) + a_2y'(0, \lambda) = a_3(\lambda - \lambda_i)y(0, \lambda), \\ y(1, \lambda) = 1. \end{gathered} \tag{9}$$

Into this problem we substitute the series (7) and the corresponding series for $y'(x,\lambda)$ and $y''(x,\lambda)$ and make use of the results of the preceding paragraph. The conditions in (9) hold for all λ and we may therefore equate coefficients of $(\lambda - \lambda_i)$ to obtain

$$\begin{gathered} A_1''(x, \lambda_i) + [\lambda_i + p(x)]A_1(x, \lambda_i) = -y_i(x), \\ a_2A_1'(0, \lambda_i) = a_3y_i(0), \\ A_1(0, \lambda_i) = 0. \end{gathered} \tag{10}$$

We multiply the differential equation in (10) by $y_i(x)$ and the differential equation $y_i'' + (\lambda_i + p)y_i = 0$ by $A_1(x,\lambda_i)$, subtract the two, integrate and obtain

$$-\int_0^1 [y_i(x)]^2\, dx = [A_1'y_i - A_1y_i']_0^1 .$$

Making use of the fact that the eigenfunction y_i satisfies the boundary conditions $U_{\lambda_i0} = U_{\lambda_i1} = 0$ we suppose that $a_2 \neq 0$ and $b_2 \neq 0$ and evaluate the right member by means of the boundary conditions in (5) and the two boundary conditions in (10). Thus we obtain

$$[y_i , y_i] - \frac{b_3}{b_2}[y_i(1)]^2 = \frac{y_i(1)}{b_2} U_{\lambda_i1}[A_1(x, \lambda_i)]. \tag{11}$$

Since the function $y(x,\lambda)$ satisfies the first boundary condition in (5) for all λ the characteristic equation of the eigenvalue problem is

$$U_{\lambda 1}[y(x, \lambda)] = 0.$$

The derivative with respect to λ of this left member when evaluated at $\lambda = \lambda_i$ is

$$U_{\lambda_i 1}[A_1(x, \lambda_i)] - b_3 y_i(1).$$

We see from (11) that if this were zero we would have $[y_i , y_i] = 0$ which is a contradiction in view of the fact that the eigenvalue problem is normal. The proof follows through with minor changes if $a_2 = 0$ or $b_2 = 0$. This completes the proof of Theorem 1.

We next define the Rayleigh's quotient $R(u)$ for any V-function $u(x)$:

$$R(u) = \frac{-[u'' + pu, u]}{[u, u]}.$$

By taking the inner product of the left member of the equation $y_i'' + (\lambda_i + p)y_i = 0$ with y_i and solving for λ_i we see that $R(y_i) = \lambda_i$; the Rayleigh's quotient for the eigenfunction $y_i(x)$ (a V-function) is the corresponding eigenvalue λ_i .

Definition 4. The problem (3), (4) shall be called "positive definite" if it is normal and if $[u'' + pu,u] < 0$ for every V-function $u(x)$.

It follows that a positive-definite eigenvalue problem has only positive eigenvalues. By performing an integration by parts we find that if $p(x) < 0$ for $0 \leqq x \leqq 1$ the conditions $a_2a_3 \leqq 0$, $b_2b_3 \geqq 0$, $a_1a_2 \leqq 0$, and $b_1b_2 \geqq 0$ guarantee that the problem is positive-definite. Other combinations of conditions which guarantee a positive-definite problem can be found.

It is easily verified that the eigenvalue problem (2) arising from the boundary value problem (1) is positive-definite because in that problem $r(x) > 0$ for $0 \leqq x \leqq 1$ since r is the thermal emissivity of the faces of the slab, and k_1 and k_2 are positive since q_1, q_2 and $1/c$ are positive since they are coefficients of conductivity of the materials involved. Hence everything that has been said and will be said concerning positive-definite problems holds for the problem (2).

4. The Green's function solution of the non-homogeneous problem. Consider the following non-homogeneous problem with the complex parameter λ:

$$\begin{aligned} y''(x, \lambda) + [\lambda + p(x)]y(x, \lambda) &= f(x), \\ U_0[y(x, \lambda)] &= 0, \\ U_1[y(x, \lambda)] &= 0, \end{aligned} \tag{12}$$

where $f(x)$ is a bounded integrable function. Except for the function $f(x)$ in the differential equation this is the eigenvalue problem (3)(4). We make the stipulation here that if in the boundary conditions $a_2 = 0 (b_2 = 0)$ then $a_3 = 0 (b_3 = 0)$. Making use of the differential equation we may rewrite the boundary conditions in the form

$$\begin{aligned} U_{\lambda 0}[y(x, \lambda)] &= -a_3 f(0), \\ U_{\lambda 1}[y(x, \lambda)] &= -b_3 f(1). \end{aligned} \tag{13}$$

It is well known (see, for example, [5] p. 257) that if λ is not an eigenvalue of (3)(4)

the solution of (12) with the boundary conditions in the form (13) is

$$y(x, \lambda) = \int_0^1 f(\xi)G(x, \xi, \lambda)\, d\xi - a_3 f(0)G_0(x, \lambda) - b_3 f(1)G_1(x, \lambda). \tag{14}$$

In this equation $G(x,\xi,\lambda)$ is the Green's function of the system and $G_0(x,\lambda)$ is the solution of the corresponding homogeneous problem except that the right member of the first condition in (13) is replaced by 1. Similarly, $G_1(x,\lambda)$ is the solution of the corresponding homogeneous problem except that the second condition in (13) is replaced by 1.

Since the Green's function satisfies the homogeneous problem we have

$$U_{\lambda 0}[G(x, \xi, \lambda)] = \lim_{x\to 0} \{a_1 - a_3[\lambda + p(0)]G(x, \xi, \lambda) \qquad 0 < \xi \leqq 1$$
$$-a_2 G_x(x, \xi, \lambda)\} = 0$$

which gives

$$U_{\lambda 0}[\lim_{\xi\to 0} G(x, \xi, \lambda)] = U_{\lambda 0}[\lim_{\xi\to 0} G(x, \xi, \lambda)] - \lim_{\xi\to 0} U_{\lambda 0}[G(x, \xi, \lambda)]$$

the last term being zero. Because G is continuous and G_x has a unit jump across the line $x = \xi$ we see from this last expression that

$$U_{\lambda 0}[\lim_{\xi\to 0} G(x, \xi, \lambda)] = a_2 .$$

Because of the properties of the Green's function the function $\lim_{\xi\to 0} G(x,\xi,\lambda)$ satisfies the homogeneous differential equation corresponding to the one in (12) and the second boundary condition in (13). The function is also continuous and has a continuous derivative. Recalling the definition of $G_1(x,\lambda)$ we see that, except for a multiplicative constant, this function must be identical with the function $\lim_{\xi\to 0} G(x,\xi,\lambda)$. More precisely, if $a_2 \neq 0$ we have

$$G_0(x, \lambda) = \lim_{\xi\to 0} \frac{G(x, \xi, \lambda)}{a_2}.$$

In case $b_2 \neq 0$ a similar argument yields

$$G_1(x, \lambda) = -\lim_{\xi\to 1} \frac{G(x, \xi, \lambda)}{b_2}.$$

Therefore in view of (14) and our definition of the inner product the solution of (12) may be written

$$y(x, \lambda) = [f(\xi), G(x, \xi, \lambda)]. \tag{15}$$

According to Collatz [3], as long as λ_i is a simple eigenvalue the Green's function has a simple pole at $\lambda = \lambda_i$ and may be written

$$G(x, \xi, \lambda) = \frac{C_i y_i(x) y_i(\xi)}{(\lambda - \lambda_i)} + G_i^*(x, \xi, \lambda),$$

where G_i^* is analytic at $\lambda = \lambda_i$ and C_i is a constant. The quantity $C_i y_i(x) y_i(\xi)$ is, therefore, the residue of G at the simple pole $\lambda = \lambda_i$. It follows that

$$\lim_{\lambda\to\lambda_i} \{(\lambda - \lambda_i)[y_i(\xi), G(x, \xi, \lambda)]\} = C_i y_i(x)[y_i(\xi), y_i(\xi)]. \tag{16}$$

Using the fact that

$$y_i''(x) + [\lambda + p(x)]y_i(x) = (\lambda - \lambda_i)y_i(x)$$

and the result (15) which gives an implicit solution for $y_i(x)$, we see that the left member in (16) is simply $y_i(x)$. Therefore

$$C_i = \frac{1}{[y_i(\xi), y_i(\xi)]}.$$

Thus the residue of G at $\lambda = \lambda_i$ is completely determined and is found to be the term of the Fourier series of $f(x)$ corresponding to the eigenvalue λ_i. If $\lambda_i (i = 0, 1, \cdots, n)$ is the set of eigenvalues inside the circle $|\lambda| = R$ the meromorphic function G may be written

$$G(x, \xi, \lambda) = \sum_{i=0}^{n} \frac{y_i(x)y_i(\xi)}{(\lambda - \lambda_i)[y_i(\xi), y_i(\xi)]} + G_R^*(x, \xi, \lambda).$$

where G_R^* is an analytic function inside the circle $|\lambda| = R$. Also, in view of (15), the solution of the non-homogeneous problem (12) may be written

$$y(x, \lambda) = \sum_{i=0}^{n} \frac{y_i(x)[f(\xi), y_i(\xi)]}{(\lambda - \lambda_i)[y_i(\xi), y_i(\xi)]} + [f(\xi), G_R^*(x, \xi, \lambda)].$$

A note concerning an expansion theorem for V-functions. With the developments of this section and sections 2 and 3, the groundwork has been laid for the proofs of the existence of an infinite number of eigenvalues and an expansion theorem for V-functions for the problem (3)(4) which is positive definite. The proofs are analogous to those given by E. Kamke [6] for problems which do not involve the parameter in the boundary conditions. Because of the analogy with Kamke's work and because the expansion theorem we shall prove in the next section is more general, the proofs are omitted.

5. An expansion theorem by the Laplace transform. In this section we shall obtain an expansion theorem by the application of the Laplace transform to a problem in heat conduction. Let us consider the transient temperatures $U(x,t)$ of a thin slab in the shape of a rectangular parallelopiped which has its upper and lower edges insulated. At the edge $x = 1$ the slab is in perfect thermal contact with a well stirred liquid the container of which is exposed to an external medium at temperature zero. The edge $x = 0$ is kept at temperature zero. If the flat faces transfer heat into the surrounding medium according to a linear law and with thermal emissivity $p(x)$, and if the initial temperature of the slab is $F(x)$, the boundary value problem may be written as

$$\begin{gathered} U_t(x, t) = U_{xx}(x, t) \neg p(x)U(x, t), \\ U(0, t) = 0, \\ K_1U(1, t) + U_x(1, t) + K_2U_t(1, t) = 0, \\ U(x, 0) = F(x), \end{gathered}$$

where the units are adjusted so that the thermal diffusivity equals 1. In this statement of the problem $K_1 = qA_2/KA_1$ and $K_2 = hM/KA_1$ and $K_2 = hM/KA_1$ where the constant q is the thermal emissivity at the end of the container, K is the coefficient of conductivity of the slab, h the specific heat of the liquid, M its mass, A_1 the area of the

slab in contact with the liquid, and A_2 the area of the container in contact with the external medium. We note therefore that $K_1 > 0$, $K_2 > 0$; also, $p(x) > 0$ for $0 \leqq x \leqq 1$. Except for the variable coefficient of heat transfer of the slab which introduces the nonconstant coefficient in the differential equation, this is essentially the problem considered by Langer [7] and Gaskell [4].

The application of the Laplace transform

$$L\{F(x)\} \equiv f(s) \equiv \int_0^\infty e^{-sx}F(x)\,dx$$

to the boundary value problem gives the following transformed problem:

$$\begin{aligned} u''(x, s) - [s + p(x)]u(x, s) &= -F(x), \\ u(0, s) &= 0, \\ (K_1 + K_2 s)u(1, s) + u'(1, s) &= K_2 F(1). \end{aligned} \tag{17}$$

We note that the homogeneous problem corresponding to this is a case of the eigenvalue problem (3)(4). From the results of Section 4 we see that the solution to this problem may be written

$$u(x, s) = -[F(\xi), G(x, \xi, s)], \tag{18}$$

where $G(x,\xi,s)$ is the Green's function for the system. From the definition of K_2 we noted that $K_2 > 0$ and hence the corresponding eigenvalue problem is normal; the problem has only real and simple eigenvalues s_n . According to the results of Section 4, $u(x,s)$ has simple poles at those eigenvalues and the residues of those poles are the terms of the Fourier series corresponding to $F(x)$.

The existence of an infinite set of eigenvalues and an expansion theorem for the eigenvalue problem can now be proved with the aid of a close examination of the order properties of various terms involved in the solution (18). In this proof we shall deal with the Laplace inversion integral

$$L_i^{-1}\{f(s)\} \equiv \lim_{\beta\to\infty} \frac{1}{2\pi i}\int_{\gamma - i\beta}^{\gamma + i\beta} e^{zt}f(z)\,dz$$

and make use of the following theorem (see [2], p. 159):

Theorem 5. Let $v(s)$ be a function of a complex variable s and of the order $O(s^{-k})$ in a half plane $\mathcal{R}(s) \geqq \gamma_0$ where $k > 1$. Then the inversion integral $L_i^{-1}\{v(s)\}$ converges along a line γ where $\gamma \geqq \gamma_0$ to a continuous function $V(t)$ which is independent of γ and such that $V(0) = 0$.

Let us now define functions $u_1(x,s)$ and $u_2(x,s)$ which are linearly independent solutions of the homogeneous differential equation corresponding to the one in (17) and are such that $u_1(0,s) = 0$ and $u_1'(0,s) = 1$, while $(K_1 + K_2 s)u_2(1,s) + u_2'(1,s) = 0$ and $u_2(1,s) = 1$. We write the initial value problem which defines $u_2(x,s)$ in the form

$$\begin{aligned} u_2''(x, s) - su_2(x, s) &= p(x)u_2(x, s), \\ u_2(1, s) &= 1, \\ u_2'(1, s) &= -(K_1 + K_2 s). \end{aligned}$$

By supposing that the right member of the differential equation is a known function, we find that the solution $u_2(x,s)$ may be written in the following implicit form:

$$u_2(x, s) = \frac{\sinh (xs^{1/2})}{\sinh (s^{1/2})} + \sinh [(1 - x)s^{1/2}]\left[\coth (s^{1/2}) + \frac{K_1 + K_2 s}{s^{1/2}}\right]$$
$$- \frac{1}{s^{1/2}} \int_x^1 \sinh [(x - \xi)s^{1/2}]p(\xi)u_2(\xi, s)\, d\xi. \tag{19}$$

It can now be shown that the function

$$M(s) = \operatorname*{Max}_{0 \le x \le 1} \left| \frac{s^{1/2} \sinh (x^{1/2})u_2(x, s) \exp [-(2 - x)s^{1/2}]}{K_1 + K_2 s} \right|$$

is bounded for R sufficiently large where $s = \mathrm{Re}^{i\theta}$. Since $M(s)$ is bounded we can see from (19) that

$$\frac{2s^{1/2} \sinh (s^{1/2})u_2(0, s) \exp (-2s^{1/2})}{K_1 + K_2 s}$$
$$= [1 - \exp (-2s^{1/2})]\{1 - \exp (-2s^{1/2}) + [1 + \exp (-2s^{1/2})]O(s^{-1/2})\} + O(s^{-1/2}). \tag{20}$$

The zeroes of the function $u_2(0,s)$ are the eigenvalues as can be seen from the definition of $u_2(x,s)$. Since these eigenvalues are real we see from expression (20) that they cannot be large and positive. Therefore we set $s = -\rho^2$ where ρ is real and find that the characteristic equation $u_2(0,s) = 0$ takes the form

$$\sin \rho\left[\sin \rho + O\left(\frac{1}{\rho}\right) \cos \rho\right] + O\left(\frac{1}{\rho}\right) = 0.$$

The applications of Rouche's theorem incomplex variable theory gives the fact that there exists an infinite number of $s_n(n = 0, 1, \cdots)$, that they are less than a fixed number γ_0, and that when n gets large $(-s_n)^{1/2}$ approaches $n\pi$. Since $u_2(0,s)$ is an analytic function it has only a finite number of zeroes in any finite region of the complex plane. Thus the existence of a denumerable set of eigenvalues is determined.

Treating the function $u_1(x,s)$ in a manner similar to that above for $u_2(x,s)$, the following result analogous to (20) can be obtained:

$$\frac{2s^{1/2}}{K_1 + K_2 s} [(K_1 + K_2 s)u_1(1, s) + u_1'(1, s)] = 1 - e^{(-2s^{1/2})} + O(s^{-1/2}) \tag{21}$$

In terms of the functions $u_1(x,s)$ and $u_2(x,s)$ the Green's function of the system (17) appearing in expression (18) may be written

$$G(x, \xi, s) = \frac{-u_2(\xi, s)u_1(x, s)}{u_2(0, s)}, \qquad 0 \leqq x < \xi,$$
$$= \frac{-u_1(\xi, s)u_2(x, s)}{u_2(0, s)}, \qquad \xi < x \leqq 1.$$

If we next make use of the properties of the Green's function and assume that $F(x)$ is continuous and has a sectionally continuous first derivative, we may integrate expression

(18) by parts to obtain

$$su(x, s) - F(x) = F(1)G_\xi(x, 1, s) - F(0)G_\xi(x, 0, s) - \int_0^1 G_\xi(x, \xi, s)F(\xi)\, d\xi$$

$$- \int_0^1 G_\xi(x, \xi, x)F(\xi)p(\xi)\, d\xi - \frac{K_2 sF(1)u_1(x, s)}{(K_1 + K_2 s)u_1(1, s) + u_1'(1, s)}.$$

Because the quantities $u_2(0,s)$ and $(K_1 + K_2s)u(1,s) + u_1'(1,s)$ appear in the denominators of the various factors and integrands above, the application of the results (20) and (21) yields

$$u(x, s) - \frac{F(x)}{s} = O(s^{-3/2}) \qquad \Re(s) \geqq \gamma_0\,,\ 0 < \epsilon \leqq x \leqq 1 - \epsilon$$

where ϵ is a fixed but arbitrary number. Thus we see that the conditions of Theorem 5 are satisfied. Let us designate the residue of $u(x,s)$ at $s = s_n$ by $R_n(x)$. If we suppose for the moment that the Laplace inversion integral of $u(x,s) - F(x)/s$ can be represented by the sum of the residues of the integrand we may write

$$L_i^{-1}\{u(x, s) - F(x)/s\} = \sum_{n=0}^{\infty} e^{s_n t}[R_n(x)] - F(x), \qquad t \geqq 0 \tag{22}$$

and therefore, according to Theorem 5,

$$\sum_{n=0}^{\infty} R_n(x) = F(x).$$

Because we have shown in Section 4 that the residues $R_n(x)$ are the terms of the Fourier series corresponding to $F(x)$, the proof of the expansion theorem rests on the proof that the equation (22) is valid, that the inversion integral can be represented by the sum of the residues of its integrand.

Since the numbers $(-s_n)^{1/2}$ approach $n\pi$ as n gets large, the parabolas P_n given by

$$r = \left(n + \frac{1}{2}\right)^2 \pi^2 \csc^2 \frac{\theta}{2} \qquad (n = 1, 2, \cdots)$$

pass between the poles of $u(x,s)$ when n is large. It can be shown that when s is a point on the parabola P_n and is sueh that $R(s) \leqq \gamma$

$$\left| u(x, s) - \frac{F(x)}{s} \right| \leq \frac{C}{(n + 1/2)^2} \tag{23}$$

for n sufficiently large where C is a positive constant; also, on the parabola P_n when θ is restricted to the range $-\pi < \theta_0 \leqq \theta \leqq \theta_0 < \pi$

$$\left| u(x, s) - \frac{F(x)}{s} \right| \leq \frac{D(\theta_0\,, \epsilon)}{(n + 1/2)^3}$$

where the positive number D, as indicated, depends on θ_0 and ϵ and gets large as θ_0 approaches π. In making these estimates we assume that $F(x)$, in addition to being continuous and having a sectionally continuous first derivative, has a sectionally continuous second integral around the closed path consisting of the parabola P_n and the line $\Re(s) = \gamma$ equals the sum of the residues of the poles enclosed. The order properties

(23) and (24) are sufficient to guarantee that the integral of $u(x,s) - F(x)/s$ over P_n goes to zero uniformly in x for $0 < \epsilon \leqq x \leqq 1 - \epsilon$ with increasing n. In other words, the inversion integral of $u(x,s) - F(x)/s$ may be represented by the sum of the residues of the integrand.

Thus we have proved that $F(x)$ may be expanded in the series of eigenfunctions of the problem

$$\begin{gathered} y''(x) - [\lambda + p(x)]y(x) = 0, \\ y(0) = 0, \\ (K_1 + K_2)y(1) + y'(1) = 0, \end{gathered} \tag{25}$$

where K_1 and K_2 are constants with $K_2 > 0$, and $p(x)$ is any continuous function on $0 \leqq x \leqq 1$. The expansion is

$$F(x) = \sum_{i=0}^{\infty} A_i y_i(x), \tag{26}$$

where

$$A_i = \frac{[F(x),\, y_i(x)]}{[y_i(x),\, y_i(x)]}$$

in which the inner product is the one corresponding to problem (25). Our main results may be summed up with the following theorem:

Theorem 6. The eigenvalues λ_i of problem (25) are real and simple, and there exists a real number γ_0 such that $\lambda_i \leqq \gamma_0$ for all λ_i and the numbers $(-\lambda_i)^{1/2}$ approach $i\pi$ as i gets large. For any continuous function $F(x)$ such that $F'(x)$ and $F''(x)$ are sectionally continuous on $0 \leqq x \leqq 1$ the expansion (26) is valid on $0 < x < 1$ and the convergence is uniform on any closed subinterval of $0 < x < 1$.

If in (25) $K_2 = 0$ the problem reduces to a standard type Sturm-Liouville problem for which expansion theorems are well known (see, for example, [2], p. 259). Although (25) is a special case of problem (3)(4), only minor changes are necessary in the proof given in this section to extend it to the more general problem.

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APPROXIMATE SOLUTION OF AN INITIAL VALUE PROBLEM BY GENERALIZED CARDINAL SERIES*

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1. Introduction. A problem which arises in many different contexts is to approximate one of a certain class of solutions, $u(x,y)$, of a partial differential equation or integral equation by a function which interpolates its values on a line. More generally, let V be a vector space, whose elements are functions of a point P in a space X, so that the sum of two functions in V belongs to V, as does the product of a function in V by a constant. Consider the problem of obtaining a function of V assuming given values a_k at points P_k, for k belonging to a set, I, of integers. If a family $\{A_k(P)\}$ of functions of V can be determined so that $A_k(P_k) = 1$, $A_k(P_j) = 0$ for $j \neq k$ $(j,k \in I)$, then such an interpolatory function is given formally by the sum $\sum_{k \in I} a_k A_k(P)$. Such functions $A_k(P)$ can sometimes be defined as follows. Let, for each point P, $\varphi(P)$ denote an element in a Hilbert space H. Let (α,β) denote the inner product of two elements, α and β, of H, and let there exist elements v_k of H such that $(\varphi(P_k),v_k) = 1$, $(\varphi(P_k),v_j) = 0$, $j \neq k$, $j,k \in I$. Then one may put $A_k(P) = (\varphi(P),v_k)$, if this inner product belongs to V. Suppose, for example, that as functions of t, $\varphi(t;P)$ and $v_k(t)$ belong to the class L_2 (have integrable square) on an interval (a,b) $(-\infty \leq a < b \leq \infty)$, that the integral $\int_a^b \varphi(t;P_j)v_k(t)\,dt$ is 1 for $j = k$ and 0 for $j \neq k$, $j,k \in I$, and that the integral belongs to V. Then one may put $A_k(P) = \int_a^b \varphi(t;P)v_k(t)dt$. If, as functions of t, the functions $\{\varphi(t;P_k)\}$ form an orthonormal system over (a,b), one may set $v_k(t) = \varphi(t;P_k)$.

We shall consider only the case where P represents a point in the xy-plane of a set E containing the x-axis, where $v_k(t) \equiv \exp(ikt)$, and where $\phi(t;x,y)$ is defined for $(x,y) \in E$ so that

$$\varphi(t;\, x,\, 0) = \exp(-ixt). \tag{1.1}$$

We define

$$A_{k,\lambda}(x,\, y) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t/\lambda;\, x,\, y) \exp(ikt)\, dt \tag{1.2}$$

for each integer k, each positive number λ, and each point (x,y) in E. The series

$$\sum_{k=-\infty}^{\infty} f(k\lambda) A_{k,\lambda}(x,y) \tag{1.3}$$

becomes, for $y = 0$,

$$\sum_{k=-\infty}^{\infty} f(k\lambda) \sin\left[\frac{\pi}{\lambda}(x - k\lambda)\right] \Big/ \frac{\pi}{\lambda}(x - k\lambda), \tag{1.4}$$

which is the cardinal series associated with values $f(k\lambda)$ at points $x = k\lambda$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$. If a function $U(x,y)$ is given for $y = 0$, $U(x,0) \equiv f(x)$, the series (1.3) gives, formally, a function defined on E coinciding with $f(x)$ at points $x = k\lambda$

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$(k = \cdots, -2, -1, 0, 1, 2, \cdots)$ on the x-axis. The advantages of the cardinal series as an interpolatory function are well-known. The cardinal series (1.4), under suitable conditions, yields a "smooth" function of x, in the sense that its Fourier transform vanishes outside the interval $(-\pi/\lambda, \pi/\lambda)$ ([1]; cf. also [2]). There is also a "consistency" property: a new cardinal series associated with values of the cardinal series (1.4) at equally-spaced points having adjacent points not farther than λ units apart coincides with the cardinal series (1.4) ([3], [4], also [2]; for further references to the literature on cardinal series see [2], also [5]). In 1908 de la Vallée Poussin proved approximation theorems [6] giving conditions sufficient in order that the series (1.4) will approach $f(x)$ as $\lambda \to 0$. It is our purpose here also to obtain an approximation theorem: to obtain conditions sufficient in order that the series (1.3) should converge, and should approach $U(x,y)$ as $\lambda \to 0$, where $U(x,y)$ is a function of V, defined on E, uniquely determined by its values $f(x)$ on the x-axis. In §3, the method and the theorem of §2 are applied to the problem of approximating the temperature at a certain instant in an infinite insulated rod in terms of its temperatures at a later instant. As is well known, this problem has also a probabilistic interpretation. If a random variable $\mathbf{z}$ having a known frequency function is known to be the sum of independent random variables, $\mathbf{x}$ and $\mathbf{y}$, the random variable $\mathbf{y}$ having a normal frequency function with mean 0 and standard deviation σ. §3 applies, and yields a method of numerical approximation to the frequency function of $\mathbf{x}$. In order for the method to apply formally, it is not necessary that $\mathbf{y}$ be normally distributed.

2. An approximation theorem. In this section standard techniques of Fourier analysis are used to prove an approximation theorem, Theorem 2.1.

We observe that the function $\sin [\pi(x - k\lambda)/\lambda]/[\pi(x - k\lambda)/\lambda]$, and the function which vanishes outside $[-\pi/\lambda, \pi/\lambda]$ and which is given by $\lambda \exp (ik\lambda t)$ on $[-\pi/\lambda, \pi/\lambda]$, are a Fourier transform pair; i.e.,

$$\frac{\sin \pi(x - k\lambda)/\lambda}{\pi(x - k\lambda)/\lambda} \equiv \frac{\lambda}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} \exp (ik\lambda t) \exp (-ixt)\, dt. \tag{2.1}$$

If the series

$$\lambda \sum_{k=-\infty}^{\infty} f(k\lambda) \exp (ik\lambda t) \tag{2.2}$$

converges uniformly, then it is clear from (2.1) that the series (1.4) converges (throughout this paper, integrals and series extending from $-\infty$ to ∞ are said to converge if the Cauchy principal value exists, and converge uniformly if $\sum_{-p}^{p}$ or $\int_{-p}^{p}$ converges uniformly as $p \to \infty$). It is desirable, therefore, to determine conditions on $f(x)$ implying that the series (2.2) converges uniformly. It is simpler to state such conditions as conditions on the transform, $F(t)$, of $f(x)$. We have the following lemma.

LEMMA 2.1: *If*

$F(t)$ is continuous and of bounded variation on $(-\infty, \infty)$, (2.3)

if, for each positive number λ,

$\sum_{j=-\infty}^{\infty} F(t + 2\pi j/\lambda)$ converges uniformly and absolutely for $|t| \leq \pi/\lambda$ to a function $F_\lambda(t)$, (2.4)

then the integral $1/2\pi \int_{-\infty}^{\infty} F(t) \exp(-ixt)dt$ *converges to a function* $f(x)$, *and the series*

$$\lambda \sum_{k=-\infty}^{\infty} f(k\lambda) \exp(ik\lambda t) \tag{2.2}$$

converges uniformly to $F_\lambda(t)$; *moreover, for* $|t| \leq \pi/\lambda$,

$$\lim F_\lambda(t) = F(t), \tag{2.5}$$

as $1/\lambda \to \infty$ *through integral values.*

Proof: We have

$$\lambda \int_{-(2p+1)\pi/\lambda}^{(2p+1)\pi/\lambda} F(t) \exp(-ik\lambda t)\, dt = \lambda \sum_{j=-p}^{p} \int_{-\pi/\lambda}^{\pi/\lambda} F(t + 2\pi j/\lambda) \exp(-ik\lambda t)\, dt$$

$$= \int_{-\pi}^{\pi} \sum_{j=-p}^{p} F(u/\lambda + 2\pi j/\lambda) \exp(-iku)\, du.$$

By hypothesis, $\sum_{j=-\infty}^{\infty} F(u/\lambda + 2\pi j/\lambda)$ converges uniformly and absolutely to $F_\lambda(u/\lambda)$ for $|u| \leq \pi$. Since λ is arbitrary (positive), it follows that $(1/2\pi) \int_{-\infty}^{\infty} F(t) \exp(-ixt)dt$ converges to a function $f(x)$, and that $\lambda f(k\lambda) = (1/2\pi) \int_{-\pi}^{\pi} F_\lambda(u/\lambda) \exp(-iku)du$. We observe that the total variation of $F_\lambda(u/\lambda)$ on the interval $|u| \leq \pi$ is not greater than the total variation of $F(t)$ on $(-\infty, \infty)$. Moreover, $F_\lambda(u/\lambda)$ is continuous (and of period 2π, as a function of u). Hence its Fourier series, $\lambda \sum_{k=-\infty}^{\infty} f(k\lambda) \exp(iku)$, converges uniformly ([7], p. 42) to $F_\lambda(u/\lambda)$, so that $\lambda \sum_{k=-\infty}^{\infty} f(k\lambda) \exp(ik\lambda t)$ converges uniformly to $F_\lambda(t)$. Hypothesis (2.4) clearly implies that, for $|t| \leq \pi/\lambda$, $\lim_{\lambda \to 0} F_\lambda(t) = F(t)$; for let t be fixed. We have $|F_\lambda(t) - F(t)| \leq \sum_{|j| \geq 1} |F(t + 2\pi j/\lambda)| \leq \sum_{|j| > N} |F(t + 2\pi j)|$ if $\lambda < 1/N$. But this latter sum is arbitrarily small for sufficiently large N. The proof of the lemma is complete.

The hypotheses on $F(t)$ of continuity and finite total variation may be replaced by others which imply the uniform convergence on $|u| \leq \pi$ of the Fourier series of $F_\lambda(u/\lambda)$. This lemma is essentially equivalent to Poisson's summation formula ([8], p. 33, ff.).

We recall that

$$\varphi(t; x, 0) = \exp(-ixt), \tag{1.1}$$

and

$$A_{k,\lambda}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t/\lambda; x, y) \exp(ikt)\, dt. \tag{1.2}$$

Using lemma 2.1, we obtain the following theorem.

THEOREM 2.1: *Let* $F(t)$ *satisfy the following conditions*:

$$F(t) \textit{ is continuous and of bounded variation on } (-\infty, \infty); \tag{2.3}$$

$$\sum_{j=-\infty}^{\infty} F(t + 2\pi j/\lambda) \text{ converges uniformly and absolutely for}$$
$$|t| \leq \pi/\lambda \textit{ to a function } F_\lambda(t); \tag{2.4}$$

to each point (x,y) *in* E *corresponds a function of* t, $L(t;x,y)$, *integrable with respect to* t *over* $(-\infty, \infty)$, *such that, for each* $\lambda > 0$,

$$|F_\lambda(t)\varphi(t;x,y)| \leq L(t;x,y), \textit{ for } |t| \leq \pi/\lambda. \tag{2.6}$$

Then the integral $(1/2\pi)\int_{-\infty}^{\infty} F(t)\exp(-ixt)dt$ *converges to a function* $f(x)$. *The series* $\sum_k f(k\lambda)A_{k,\lambda}(x,y)$ *converges to the function* $f_\lambda(x,y) \equiv (1/2\pi)\int_{-\pi/\lambda}^{\pi/\lambda} F_\lambda(t)\varphi(t;x,y)dt$ *for* $(x,y) \in E$; $f_\lambda(x,0)$ *is the cardinal series associated with values* $f(k\lambda)$, *so that in particular,* $f_\lambda(k\lambda,0) = f(k\lambda)$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$. *Moreover, as* $1/\lambda \to \infty$ *through integral values,* $f_\lambda(x,y) \to f(x,y)$ *for* (x,y) *in* E, *where* $f(x,y) = (1/2\pi)\int_{-\infty}^{\infty} F(t)\varphi(t;x,y)\,dt$, *a function coinciding with* $f(x)$ *on the* x*-axis.**

Proof: That the integral $(1/2\pi)\int_{-\infty}^{\infty} F(t)\exp(-ixt)dt$ converges, is guaranteed by lemma 2.1. Also, by lemma 2.1, the series $\lambda\sum_{k=-\infty}^{\infty} f(k\lambda)\exp(ik\lambda t)$ converges uniformly for $|t| \le \pi/\lambda$ to $F_\lambda(t)$. We have

$$\sum_{k=-n}^{n} f(k\lambda)A_{k,\lambda}(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} f(k\lambda)\exp(ikt)\varphi(t/\lambda; x, y)\,dt,$$

$$= \frac{\lambda}{2\pi}\int_{-\pi/\lambda}^{\pi/\lambda}\sum_{k=-n}^{n} f(k\lambda)\exp(ik\lambda u)\varphi(u; x, y)\,du.$$

Hence

$$\sum_{k=-n}^{n} f(k\lambda)A_{k,\lambda}(x, y) \qquad \text{converges to} \qquad f_\lambda(x, y), \tag{2.7}$$

where

$$f_\lambda(x, y) = \frac{1}{2\pi}\int_{-\pi/\lambda}^{\pi/\lambda} F_\lambda(t)\varphi(t; x, y)\,dt. \tag{2.8}$$

By (1.4), $f_\lambda(x,0)$ is the cardinal series assuming values $f(k\lambda)$ at points $x = k\lambda$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$. By (2.5), hypothesis (2.6), and Lebesgue's bounded convergence theorem, we have

$$\lim_{\lambda\to 0} f_\lambda(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\varphi(t; x, y)\,dt = f(x, y),$$

which completes the proof of the theorem.

Essentially, Theorem 2.1 gives conditions sufficient in order that an integral of the form $\int_{-\infty}^{\infty} F(t)\varphi(t)dt = \int_{-\infty}^{\infty}\varphi(t)dt\int_{-\infty}^{\infty} f(x)\exp(ixt)dx$ can be approximated by replacing $f(x)$ by the cardinal series associated with its values at points $x = k\lambda$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$, $\lambda > 0$. The parameters (x,y) are mentioned explicitly in the above formulation with a view to the application in which it is desired to continue into a set E of the xy-plane a function given on the x-axis, which belongs to a certain class.

3. Temperature on an infinite insulated rod. Let a be a positive constant, and let $U(x,y)$ denote the temperature at the point with coordinate x on an infinite, insulated rod, at time $y \ge -a$. It is known that if $U(x, -a)$ is piecewise continuous, bounded,

**Note added in proof:* The hypothesis in Theorems 2.1 and 3.1 that $F(t)$ is continuous and of bounded variation on $(-\infty, \infty)$ serves only to justify the term-by-term integration of the product of $\varphi(t/\lambda; x, y)$ by the Fourier series of $F_\lambda(t)$. Accordingly, hypothesis (2.3) in Theorem 2.1 may be replaced by the weaker hypothesis that $F_\lambda(t)$ is integrable over $[-\pi/\lambda, \pi/\lambda]$, if the further hypothesis that $\varphi(t/\lambda; x, y]$ is of bounded variation as a function of t $(x, y, \lambda$ being fixed) on $[-\pi, \pi]$ is added. Since these hypotheses are satisfied in the special case to which Theorem 3.1 applies, in that theorem the hypotheses of continuity and bounded variation on $F(t)$ may be omitted. Correspondingly, in Corollary 3.1 the hypothesis that $x^\alpha f(x)$ is absolutely integrable over $(-\infty, \infty)$ for some $\alpha > 1$ may be replaced by the hypothesis that $f(x)$ is absolutely integrable over $(-\infty, \infty)$.

and satisfies a Lipschitz condition, then the function $U(x,y)$ given by

$$U(x, y) = \frac{1}{2[\pi(y + a)]^{1/2}} \int_{-\infty}^{\infty} U(\xi, -a) \exp \{-(\xi - x)^2/4(y + a)\} \, d\xi$$

is the unique solution of the heat equation for $y > -a$ which, together with its partial derivative with respect to x is bounded ($| U(x,y) | \leq M$, $-\infty < x < \infty$, $y > -a$, $| U_x(x,y) | \leq M_2$, $| x | \geq x_1$, $y > -a$) and which approaches $U(x_0, -a)$ at a point of continuity as (x,y) approaches $(x_0, -a)$ from above ($y > -a$).

Let V be the class of functions $u(x,y)$, defined for $y \geq -a$, such that

$$u(x, y) = \frac{1}{2[\pi(y + a)]^{1/2}} \int_{-\infty}^{\infty} u(\xi, -a) \exp \{-(\xi - x)^2 /4(y + a)\} \, d\xi \tag{3.1}$$

for $y > -a$. We take for the set E the set of all points (x,y) with $y > -a$. Let us suppose that $U(x,y) \varepsilon V$, that $U(x,0) \equiv f(x)$ is known, that $U(x, -a)$ is absolutely integrable over $(-\infty, \infty)$, and that it is desired to express $U(x,y)$ for $y > -a$ in terms of $f(x)$. We observe that for fixed $y > -a$, $U(x,y)$ is the convolution of the functions $U(x, -a)$ and $\exp \{-x^2/4(y + a)\}/2[\pi(y + a)]^{1/2}$. Thus if $V(t,y) = \int_{-\infty}^{\infty} U(x,y) \exp (ixt)dx$, we have

$$V(t, y) \equiv V(t, -a) \exp \{-t^2(y + a)\} \qquad \text{for} \qquad y > -a. \tag{3.2}$$

Hence

$$V(t, 0) = V(t, -a) \exp (-at^2), \tag{3.3}$$

and

$$V(t, y) = V(t, 0) \exp (-yt^2) \qquad \text{for} \qquad y > -a. \tag{3.4}$$

Since $U(x,y) = (1/2\pi) \int_{-\infty}^{\infty} V(t,y) \exp (-itx)dt$, we have

$$U(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp (-itx - yt^2) \, dt \int_{-\infty}^{\infty} U(\xi, 0) \exp (i\xi t) \, d\xi \tag{3.5}$$

for $y > -a$. This formula gives $U(x,y)$ in terms of $U(x,0) = f(x)$, but in a form unsuitable for numerical approximation. If $y > 0$, we may interchange the order of integration in (3.5), or use (3.4) directly, to obtain

$$U(x, y) = \frac{1}{2(\pi y)^{1/2}} \int_{-\infty}^{\infty} U(\xi, 0) \exp \{-(\xi - x)^2/4y\} \, d\xi, \tag{3.6}$$

but this formula is not available for $y < 0$. The method developed in §1 and §2 applies, however, and yields an interpolation formula,

$$f_\lambda(x, y) = \sum_{k=-\infty}^{\infty} f(k\lambda) A_{k,\lambda}(x, y), \tag{3.7}$$

which, under suitable conditions, approximates $U(x,y)$ for $y > -a$.

Let

$$\varphi(t; x, y) \equiv \exp (-itx - yt^2), \tag{3.8}$$

and

$$A_{k,\lambda}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t/\lambda; x, y) \exp (ikt) \, dt. \tag{1.2}$$

We have

$$A_{k,\lambda}(x, y) = \Phi(x/\lambda - k, y/\lambda^2), \tag{3.9}$$

where

$$\Phi(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-itx - yt^2)\, dt. \tag{3.10}$$

Applying Theorem 2.1, we obtain the following theorem.

THEOREM 3.1: *Let $U(x,y)$ be a function of the class V; i.e., let*

$$U(x, y) = \frac{1}{2[\pi(y + a)]^{1/2}}\int_{-\infty}^{\infty} U(\xi, -a) \exp\{-(\xi - x)^2/4(y + a)\}\, d\xi \quad \textit{for} \quad y > -a.$$

Let $U(x,-a)$ be absolutely integrable over $(-\infty, \infty)$. Let $F(t) \equiv V(t,0) \equiv \int_{-\infty}^{\infty} f(x) \exp(ixt)dt$, where $f(x) \equiv U(x,0)$, and suppose that $F(t)$ is continuous, and of bounded variation on $(-\infty, \infty)$. Then the series $\sum_{k=-\infty}^{\infty} f(k\lambda)\Phi(x/\lambda - k,y/\lambda^2)$ converges to a function $f_\lambda(x,y)$ of V such that $f_\lambda(x,0)$ is the cardinal series associated with values $f(k\lambda)$, so that in particular $f_\lambda(k\lambda,0) = f(k\lambda)$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$. Moreover, as $1/\lambda \to \infty$ through integral values, $f_\lambda(x,y)$ approaches $U(x,y)$ $(y > -a)$.

Thus, if a sufficiently small unit interval is chosen on the x-axis, the formula $U(x,y) \simeq f_1(x,y) = \sum_{k=-\infty}^{\infty} f(k)\Phi(x - k,y)$ provides a method of numerical integration of (3.5), for negative as well as for positive values of y.

Proof: The above discussion, leading to equations (3.2) and (3.3), shows that $F(t) \equiv V(t,0)$ exists. Since $U(x,-a)$ is absolutely integrable over $(-\infty, \infty)$, the function $V(t,-a)$ is bounded. Equation (3.3) then insures that the series $\sum_{j=-\infty}^{\infty} F(t + 2\pi j/\lambda)$ converges unifirmly and absolutely for $|t| \leq \pi/\lambda$ to a function $F_\lambda(t)$ which is $0(\exp(-at^2))$. Hence, for $y > -a$, $F_\lambda(t)\varphi(t;x,y) = 0(\exp\{-(y + a)t^2\})$. Thus the hypotheses of Theorem 2.1 are satisfied, the set E being the set of all points (x,y) for which $y > -a$. We conclude that as $\lambda \to 0$, $f_\lambda(x,y) \to f(x,y) = (1/2\pi) \int_{-\infty}^{\infty} F(t)\varphi(t;x,y)dt$; but by (3.5) this is just $U(x,y)$. It remains to show that $f_\lambda(x,y) \in V$. One verifies easily that

$$\varphi(t; x, y) = \frac{1}{2[\pi(y + a)]^{1/2}}\int_{-\infty}^{\infty} \varphi(t; \xi, -a) \exp\{-(\xi - x)^2/4(y + a)\}\, d\xi \tag{3.11}$$

for $y > -a$, i.e., $\phi(t;x,y) \in V$ for each t. By Theorem 2.1, and (3.11),

$$\begin{aligned} f_\lambda(x, y) &= \frac{1}{2\pi}\int_{-\pi/\lambda}^{\pi/\lambda} F_\lambda(t)\varphi(t; x, y)\, dt \\ &= \frac{1}{4\pi^{3/2}(y + a)^{1/2}}\int_{-\pi/\lambda}^{\pi/\lambda} \exp(at^2)F_\lambda(t)\, dt \int_{-\infty}^{\infty} \exp\{-i\xi t - (\xi - x)^2/4(y + a)\}\, d\xi \\ &= \frac{1}{2[\pi(y + a)]^{1/2}}\int_{-\infty}^{\infty} \exp\{-(\xi - x)^2/4(y + a)\}\, d\xi\left[\frac{1}{2\pi}\int_{-\pi/\lambda}^{\pi/\lambda} F_\lambda(t)\varphi(t; \xi. -a)\, dt\right] \end{aligned}$$

for $y > -a$, the interchange of integrals being justified by virtue of the uniformity with respect to t of the convergence of the integral

$$\int_{-\infty}^{\infty} \exp\{-i\xi t - (\xi - x)^2/4(y + a)\}\, d\xi.$$

But

$$f_\lambda(\xi, -a) = \frac{1}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} F_\lambda(t)\varphi(t; \xi, -a)\, dt,$$

hence

$$f_\lambda(x, y) = \frac{1}{2[\pi(y + a)]^{1/2}} \int_{-\infty}^{\infty} f_\lambda(\xi, -a) \exp\{-(\xi - x)^2/4(y + a)\}\, d\xi \qquad (y > -a),$$

i.e., $f_\lambda(x,y) \in V$. This completes the proof of the theorem.

It seems desirable to state conditions on f alone, so far as is possible, sufficient for the conclusion of Theorem 3.1. To this end we prove the following lemma.

LEMMA 3.1: *If $x^\alpha f(x)$ is absolutely integrable over $(-\infty, \infty)$, $\alpha \geq 0$, and if $\int_{-\infty}^{\infty} U((2a)^{1/2}v + x, -a) \exp(-v^2/2)dv$ converges uniformly with respect to x (in particular if $U(x,-a)$ is bounded), then $x^\alpha U(x,-a)$ is absolutely integrable over $(-\infty, \infty)$.*

Proof: Suppose the contrary; then if U_+ and U_- denote the positive and negative parts of $U(x,-a)$ respectively, either $x^\alpha U_+(x,-a)$ fails to be integrable over $(0,\infty)$, or over $(-\infty,0)$, or $x^\alpha U_-(x,-a)$ so fails. Suppose the first eventuality occurs. We have

$$f_+(x) = \frac{1}{2(\pi a)^{1/2}} \int_{-\infty}^{\infty} U_+(\xi, -a) \exp\{-(\xi - x)^2/4a\}\, d\xi$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} U_+([2a]^{1/2}v + x, -a) \exp(-v^2/2)\, dv,$$

$$\int_N^M |x|^\alpha f_+(x)\, dx = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(-v^2/2)\, dv \int_{N+(2a)^{1/2}v}^{M+(2a)^{1/2}v} |t - (2a)^{1/2}v|^\alpha U_+(t, -a)\, dt$$

$$> \frac{1}{(2\pi)^{1/2}} \int_0^{1/(2a)^{1/2}} \exp(-v^2/2)\, dv \int_{N+1}^{M} (t - 1)^\alpha U_+(t, -a)\, dt$$

If $N > 0$. Now $(t - 1)^\alpha = t^\alpha(1 - 1/t)^\alpha \geq (1/2^\alpha)t^\alpha$ on $[N + 1, M]$ if $N > 1$, hence

$$\int_N^M x^\alpha f_+(x)\, dx > \frac{1}{2^\alpha} Erf\,[1/(2a)^{1/2}] \int_{N+1}^{M} t^\alpha U_+(t, -a)\, dt, \text{ where}$$

$Erf(z) = 1/(2\pi)^{1/2} \int_0^z \exp(-v^2/2)dv$. Since $\int_{N+1}^M t^\alpha U_+(t,-a)dt$ is arbitrarily large for proper choice of N, M, $x^\alpha f_+(x)$ is not integrable, contrary to the hypothesis. We obtain similarly a contradiction if $x^\alpha U_+$ is not integrable over $(-\infty,0)$, or if $x^\alpha U_-$ is not integrable over $(0,\infty)$, or over $(-\infty,0)$.

In connection with the hypotheses of Theorem 3.1, we now make the following observations:

(A) *If $\int_{-\infty}^{\infty} U[(2a)^{1/2}v + x,-a] \exp(-v^2/2)dv$ converges uniformly with respect to x (in particular, if $U(x,-a)$ is bounded), and if $f(x) \equiv U(x,0)$ is absolutely integrable, so also is $U(x,-a)$.* This is an immediate consequence of Lemma 3.1, with $\alpha = 0$.

(B) *If $f(x)$ is absolutely integrable over $(-\infty, \infty)$, then $F(t) = \int_{-\infty}^{\infty} f(x) \exp(ixt)dx$ is continuous.* The proof is immediate.

(C) *If $U(x,-a) \sum_{n=-\infty}^{\infty} \exp\{-(an\pi/x)^2\}$ is absolutely integrable, then $F(t)$ is of bounded variation on $(-\infty, \infty)$.*

Proof: By (3.3), we have $F(t) = V(t,-a) \exp(-at^2)$, where $V(t,-a) = \int_{-\infty}^{\infty} U(x,-a) \exp(ixt)dx$. Hence

$$\sum_{i=M}^{N} | F_c(t_i'') - F_c(t_i') | = \sum_{i=M}^{N} | V_c(t_i'', -a) \exp(-at_i''^2) - V_c(t_i', -a) \exp(-at_i'^2) |,$$

where the subscript c (for cosine transform) denotes the real part of the corresponding function. This latter sum is given by

$$\sum_{i=M}^{N} \left| \int_{-\infty}^{\infty} U(x, -a)[\exp(-at_i''^2) \cos xt_i'' - \exp(-at_i'^2) \cos xt_i'] \, dx \right|,$$

which is not greater than

$$\int_{-\infty}^{\infty} | U(x, -a) | \sum_{i=M}^{N} | \exp(-at_i''^2) \cos xt_i'' - \exp(-at_i'^2) \cos xt_i' | \, dx.$$

Now the total variation of $\exp(-at^2) \cos xt$ on $(-\infty, \infty)$ is given by $2 \sum_{n=-\infty}^{\infty} \exp\{-a(n\pi/x)^2\}$, hence the right hand member is not greater in absolute value than K, where $K = \int_{-\infty}^{\infty} | U(x,-a) | \sum_{n=-\infty}^{\infty} \exp\{-a(n\pi/x)^2\}dx$. Thus $F_c(t)$ is of bounded variation on $(-\infty, \infty)$. For $F_s(t)$ we replace $\cos xt$ by $\sin xt$. The total variation of $\exp(-at^2) \sin xt$ on $(-\infty, \infty)$ is given by $2 \sum_{n=-\infty}^{\infty} \exp\{-a[(n + \frac{1}{2})\pi/x]^2\}$. But $\sum_{n=0}^{\infty} \exp\{-a[(n + \frac{1}{2})\pi/x]^2\} < \sum_{n=0}^{\infty} \exp\{-a(n\pi/x)^2\}$, and $\sum_{n=-\infty}^{-1} \exp\{-a[(n + \frac{1}{2})\pi/x]^2\} < \sum_{n=-\infty}^{0} \exp\{-a[(n + \frac{1}{2})\pi/x]^2\} = 1 + \sum_{n=0}^{\infty} \exp\{-a[(n + \frac{1}{2})\pi/x]^2\} < 1 + \sum_{n=0}^{\infty} \exp\{-a(n\pi/x)^2\}$. Hence $U(x,-a) \sum_{N=-\infty}^{\infty} \exp\{-a[(n + \frac{1}{2})\pi/x]^2\}$ is also absolutely integrable over $(-\infty, \infty)$, so that $F_s(t)$ is of bounded variation on $(-\infty, \infty)$ also.

(D) *If there exists a number $\alpha > 1$ such that $x^\alpha f(x)$ is absolutely integrable over $(-\infty, \infty)$, and if $\int_{-\infty}^{\infty} U((2a)^{1/2}v + x,-a) \exp(-v^2/2)dv$ converges uniformly with respect to x (in particular, if $U(x,-a)$ is bounded), then $F(t)$ is of bounded variation on $(-\infty, \infty)$.*

Proof: By lemma 3.1, $x^\alpha U(x,-a)$ is absolutely integrable over $(-\infty, \infty)$. But $\sum_{n=-\infty}^{\infty} \exp\{-a(n\pi/x)^2\} = o(x^\alpha)$ as $x \to \infty$, for $\alpha > 1$. To see this, we have only to observe that $\max_x x^{-b} \exp\{-a(n\pi/x)^2\} = \exp(-b/2) (b/2a\pi^2)^{b/2}(1/n)^b$. Hence $x^{-b} \sum_{n=-\infty}^{\infty} \exp\{-a(n\pi/x)^2\} < x^{-b} + 2 \exp(-b/2) (b/2a\pi^2)^{b/2} \sum_{n=1}^{\infty} 1/n^b$. On choosing b between 1 and α, it becomes clear that $x^{-\alpha} \sum_{n=-\infty}^{\infty} \exp\{-a(n\pi/x)^2\} \to 0$ as $x \to \infty$. Thus the hypotheses of (C) are satisfied.

These observations yield the following corrollary of Theorem 3.1:

COROLLARY 3.1: *If $U(x,y) \in V$, i.e., if*

$$U(x, y) = \frac{1}{2[\pi(y + a)]^{1/2}} \int_{-\infty}^{\infty} U(\xi, -a) \exp\{-(\xi - x)^2/4(y + a)\} \qquad for \qquad y > -a,$$

if $U(x,-a)$ is bounded, and if $x^\alpha f(x)$ is absolutely integrable over $(-\infty, \infty)$ for some number $\alpha > 1$, where $f(x) \equiv U(x,0)$, then the series $\sum_{k=-\infty}^{\infty} f(k\lambda)\Phi(x/\lambda - k,y/\lambda^2)$ converges to a function $f_\lambda(x,y)$ of V such that $f_\lambda(x,0)$ is the cardinal series associated with values $f(k\lambda)$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$. Mbreover, $f_\lambda(x,y) \to U(x,y)$ $(y > -a)$, as $1/\lambda \to \infty$ through integral values.

4. A probabilistic interpretation. It is convenient to perform a translation of the xy-plane parallel to the y-axis, so that the point $(0, -y_0)$ $(0 < y_0 < a)$ becomes the new origin. Let $f^*(x)$ denote the function $U(x, -y_0)$, and $h^*(x)$ the function $U(x,0)$. Formula (3.4) yields

$$U(x, 0) = \frac{1}{2(\pi y_0)^{1/2}} \int_{-\infty}^{\infty} U(\xi, -y_0) \exp\{-(\xi - x)^2/4y_0\}\, d\xi, \qquad \text{or}$$

$$h^*(x) = \frac{1}{(2\pi)^{1/2}\sigma} \int_{-\infty}^{\infty} f^*(\xi) \exp\{-(\xi - x)^2/4y_0\}\, d\xi. \tag{4.1}$$

If h^* is regarded as known, and f^* as unknown, §3 gives a method of approximate solution of the integral equation (4.1).

Put $\sigma = (2y_0)^{1/2}$; we have

$$h^*(x) = \frac{1}{(2\pi)^{1/2}\sigma} \int_{-\infty}^{\infty} f^*(\xi) \exp\{-(\xi - x)^2/2\sigma^2\}\, d\xi,$$

so that $h^*(x)$ is the expected value of a function $f^*(\mathbf{w})$ expressed in terms of the mean, x, of the normally distributed random variable $\mathbf{w}$ with standard deviation σ. Thus §3 provides a means of approximating a function $f(x)$, if the expected value of the function $f(\mathbf{w})$ is known in terms of the mean of the normally distributed random variable $\mathbf{w}$ having known standard deviation σ.

If $f(x)$ is a probability frequency function, then $h^*(x)$ is the probability frequency function of the sum of two independent random variables, one of which has the frequency function $f(x)$, while the other is normally distributed with mean 0 and standard deviation σ. Thus if it is known that a random variable $\mathbf{z}$ having a known distribution is the sum of two independent random variables, $\mathbf{x}$, and $\mathbf{y}$, $\mathbf{y}$ being normally distributed with mean 0 and standard deviation σ, the method of §3 can be used to approximate the frequency function of $\mathbf{x}$.

It is not necessary that $\mathbf{y}$ be normally distributed in order to apply the method formally. Let $\mathbf{x}$ have the unknown frequency function $f^*(x)$, $\mathbf{y}$ the known frequency function $g^*(y)$; let $\mathbf{x}$ and $\mathbf{y}$ be independent, and let $\mathbf{z} = \mathbf{x} + \mathbf{y}$ have the known frequency function $h^*(z)$. Then $h^*(z) = \int_{-\infty}^{\infty} f^*(x)g^*(z - x)dx$. Let $H(t)$ denote the transform of h^*, $F(t)$ the transform of f^*, and $G(t)$ the transform of g^*. Formally, we have $H(t) = F(t)G(t)$, $F(t) = H(t)/G(t)$, and $f^*(x) = (1/2\pi) \int_{-\infty}^{\infty} \exp(-ixt)/G(t)dt \int_{-\infty}^{\infty} h^*(\xi) \exp(i\xi t)d\xi$. Now replace $h^*(\xi)$ by the cardinal series associated with its values at points $k\lambda(\lambda > 0,\ k \text{ integer})$:

$$h_\lambda(\xi) = \sum_{k=-\infty}^{\infty} h(k\lambda) \sin\frac{\pi}{\lambda}(x - k\lambda) \Big/ \frac{\pi}{\lambda}(x - k\lambda).$$

We obtain

$$f_\lambda(x) = \sum_{k=-\infty}^{\infty} h^*(k\lambda) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{G(t/\lambda)} \exp(-ixt/\lambda) \exp(ikt)\, dt. \tag{4.1}$$

Thus if we put $\varphi(t;x,y) = \exp(-ixt)$ for $y = 0$, $\varphi(t;x,y) = \exp(-ixt)/G(t)$ for $y = y_0$, and let E be the set consisting of the lines $y = 0$, $y = y_0$, then equation (4.1) is the result of a formal application of the method to which Theorem 2.1 applies. However, it may be difficult, in individual cases, to verify the hypotheses of the theorem.

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AN ASYMMETRICAL FINITE DIFFERENCE NETWORK*

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Introduction. Finite difference techniques have been used extensively in recent years in the solution of two-dimensional second order boundary value problems that have proved to be intractable by other methods. The differential equation is replaced by a system of linear algebraic equations, the solution of which gives the values of the wanted function at a finite number of points lying at the intersections of a gridwork. The use of regular polygons, either squares or equilateral triangles, in the formation of these gridworks has the desirable property that the equations associated with each node (intersection) point have a particularly simple, symmetrical form that is identical for all interior points. There are, however, two troublesome problems which arise in connection with the use of regular polygons. The first of these arises when the region has curved boundaries. In such cases some node points near the boundary will be connected to the boundary by gridwork elements of irregular lengths, necessitating the use of special equations for these points. The second problem concerns the change of mesh size at points within the boundary. It is frequently uneconomical from the point of view of the labor of computation to use the same mesh size at all points. In the neighborhood of a sharp corner or near other types of singularities, the mesh size must be reduced if an accuracy is to be obtained that is comparable with the accuracy of the solution in parts of the region where the behaviour of the wanted function is more uniform. Both of these problems have received attention from writers on relaxation methods and it is with these problems that the present paper is principally concerned. A method will be described by means of which the coefficients of the system of algebraic equations can be computed for an arbitrary distribution of node points. The positions of these node points can then be chosen to fit the boundary conditions and other special requirements of each problem.

In the construction of a finite difference gridwork to be used in the solution of physical problems, it is helpful to associate physical properties with the elements of the gridwork. Southwell and his co-workers have regarded the gridwork as a network of tensioned strings [1] while others have regarded the gridwork as a network of electrical elements [2]. In the case of a second order boundary value problem, this clear physical picture of the gridwork is lost if the differential equation is replaced by difference equations involving difference operators higher than the second order. In order to preserve the physical picture and to simplify the calculations, the higher order difference terms are usually regarded as corrections which are added in the final stage of calculation, if the relaxation method is to be used [3].

Another important reason for eliminating higher order difference operators arises in connection with analog computing devices, in which the physical picture of the network is realized. Electrical circuits have been used extensively in the solution of many kinds of boundary value problems [4,5,6]. In the construction of these circuits one consideration enters that is not present when the finite-difference equations are solved by purely numerical methods, namely that the circuit must be physically realizable. If the

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circuit is to be constructed of resistors only, it must contain no "negative" resistors, and the resistors must have the same resistance looked at from either end. In terms of the matrix of coefficients of the finite difference equations, the necessary and sufficient conditions that a network of resistors satisfying the equations is physically realizable are that the matrix must be symmetrical; that the nondiagonal terms in any row must all have a sign opposite to that of the diagonal term; and that the absolute value of their sum must be less than the absolute value of the diagonal term. These assertions can be easily verified by an examination of the equations of a network of resistors. Such restrictions are not imposed on purely numerical solutions of the difference equations. Since the author of this paper is primarily interested in the solution of boundary value problems by means of electrical analogy, these restrictions have been imposed on the methods to be presented. These restrictions will have the effect of eliminating difference operators of higher than the second order in the equations for the network.

The problem of changing cell size within a given rectangular gridwork has been solved by Southwell (Ref. 1, pp. 98–100) by a method which leads to network elements which are "physically unrealizable" according to the rules laid down above. A network for the solution of Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{1}$$

is shown in Fig. 1. The finite difference equivalent of Laplace's equation for non-ex-

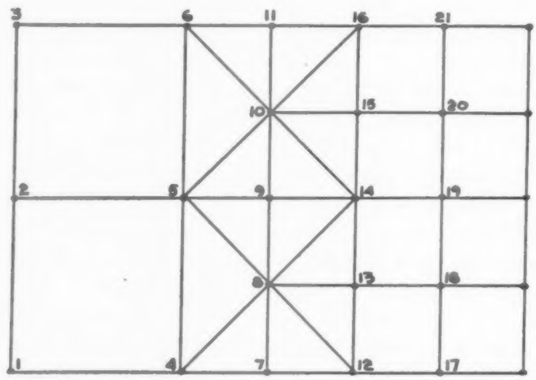

FIG. 1. Network for changing cell-size in a network of squares by Southwell's method.

ceptional points in a square gridwork is, neglecting fourth order and higher difference operators,

$$\sum_{n=1}^{4} (\phi_n - \phi_0) = 0, \tag{2}$$

where ϕ_0 is the value of the function at the point in question, and ϕ_n is the value of the function at one of the nearest neighbors of the point in question.

This equation is made to apply to exceptional points by placing upon it the following interpretation: the value of the wanted function at any point is equal to the average of

its values at the vertices of a square, the center of which is at the point in question. For example in the network of Fig. 1:

$$\begin{aligned} \phi_{14} + \phi_6 + \phi_2 + \phi_4 - 4\phi_5 &= 0, \\ \phi_{14} + \phi_{16} + \phi_6 + \phi_5 - 4\phi_{10} &= 0, \\ \phi_{14} + \phi_{10} + \phi_5 + \phi_8 - 4\phi_9 &= 0. \end{aligned} \tag{3}$$

The network resulting from these equations is physically unrealizable because the matrix of coefficients is not symmetrical. For instance, the coefficient of ϕ_{10} in the third equation is $+1$ but the coefficient of ϕ_9 in the second equation is zero.

The treatment that has been given to the problem of a curved boundary by writers on relaxation methods is largely intuitive (Ref. 1, pp. 67–78). From a consideration of the equilibrium of his tensioned network of strings Southwell concludes that, in the neighborhood of a boundary where the wanted function is known, the tension of a string connecting an interior point to the boundary should be inversely proportional to the length of the string. As applied to Eq. (2) this means that each term for which ϕ_n normally would fall outside the boundary should be weighted inversely as the length of the distance between ϕ_0 and the boundary.

The treatment of boundaries along which the normal derivative of the function is specified is less simple. In terms of the physical model, the transverse load at the edge is replaced by a statically equivalent set of forces applied to nodes just inside the boundary and to "ficticious" nodes just outside the boundary as in Fig. 2. The coefficients of the terms of Eq. (2) involving "ficticious" nodes have values between zero and one. In some instances the coefficient for an element between "ficticious" nodes (e.g. the element between nodes 2 and 3 of Fig. 2) is set equal to zero and in other instances it is set equal

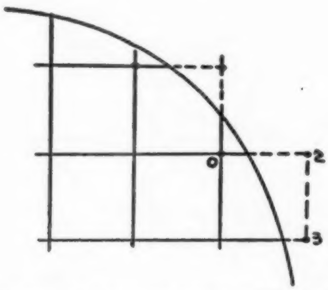

Fig. 2. Network of squares near a curved boundary.

to 1/2. In a recent article, which was largely concerned with the development of accurate network formulas, the conclusion was reached that "the writer has failed to find any completely satisfactory method of dealing accurately with boundary conditions [involving a derivative] when the direction of the normal cannot be identified with that of a mesh line as in the case of curved boundaries" [7].

The difficulty with a curved boundary is caused by the fact that for a network of regular polygons the location of node points is unalterable. By using irregular polygons it will be possible to put the node points on the boundary. In this paper the situation

will be further clarified by giving a precise physical significance, in terms of the original field problem, to the terms of the resulting generalized difference equations.

Derivation of the asymmetrical network. The problem at hand is the solution of the following equation together with appropriate conditions on ϕ and its normal derivative at boundary points of a finite plane region.

$$\nabla \cdot (\sigma \nabla \phi) + \tau = 0. \tag{4}$$

The following physical interpretation may be made of the symbols appearing in this equation: ϕ is the electrical potential in a plane region of conducting material. σ is the conductivity of this material. τ is the density of currents inserted into the region from external sources. $-\sigma\nabla\phi$ is the vector density of currents flowing in the material. σ and τ may be scalar functions of position and τ may also be a linear function of ϕ.

This physical interpretation will aid in visualizing the constructions to be made.

The problem of forming an asymmetrical network whose equations will replace Eq. (4) can be stated in the following manner. Given a region in which Eq. (4) holds and a large number of points in the region chosen at random, in what way should the points be interconnected with "physically realizable" electrical resistors in order that the voltages at the nodes shall be as nearly as possible the correct solutions of the boundary value problem characterized by Eq. (4) and appropriate boundary conditions?

A unique answer cannot be given to this question at this time. A reasonable necessary condition that should be applied to the network is that for a homogeneous conductivity (σ = constant) and a uniform field ($\nabla\phi$ = constant), which is however arbitrarily oriented, the voltage at the nodes should give the exact solution of Eq. (4). It will be shown that more than one network connecting the given points can be constructed that satisfies this condition.

In the method of solution that has been chosen the first step is to connect the randomly chosen points by a network of triangles, as in Fig. 3. The network should be planar (no

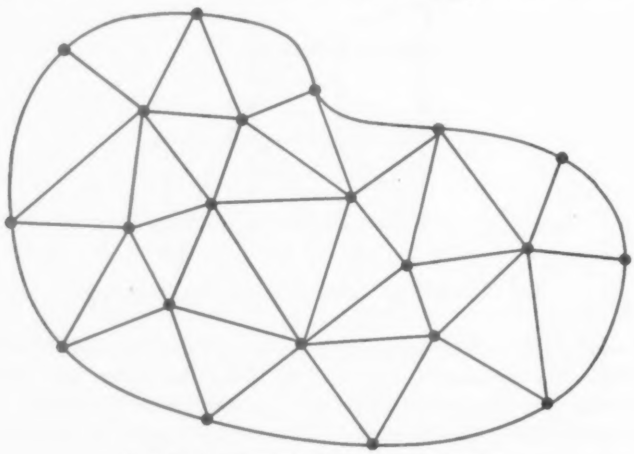

FIG. 3. Asymmetrical network of triangles.

cross-overs) and none of the interior angles of the triangles should be obtuse. It may be necessary to insert a few additional points in order to fulfill the last condition.

Consider a portion of this network shown in Fig. 4. The perpendicular bisectors of the sides of the triangles divide the region into polygons surrounding each point. A network of resistors is now constructed connecting the vertices of the triangles. The voltage across each resistor shall be interpreted as the line integral of the gradient of the potential between the two points it connects. For example, for points A and B of Fig. 4:

$$V_B - V_A = \int_A^B \nabla\phi \cdot \mathbf{dl}. \tag{5}$$

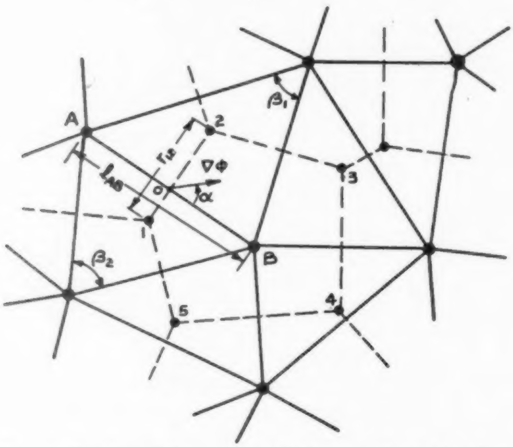

FIG. 4. Portion of the asymmetrical network of triangles.

The current in the resistor shall be interpreted as the total normal flux crossing the common boundary of the dotted polygons surrounding the two points. Since the current density is $-\sigma\nabla\phi$, we have:

$$I_{AB} = -\int_1^2 \sigma(\nabla\phi \cdot \mathbf{n})\, dr, \tag{6}$$

where $\mathbf{n}$ is a unit vector normal to dr. If $\nabla\phi$ and $\sigma\nabla\phi$ are now expanded in Taylor's series about the point 0, the midpoint of the segment $A-B$, and all terms except the first are neglected, $\nabla\phi \cong (\nabla\phi)_0$ and $\sigma\nabla\phi \cong \sigma_0(\nabla\phi)_0$. Since the segment AB is normal to the segment 1-2, the projection of $\nabla\phi$ on $\mathbf{dl}$ is the same as the projection of $\nabla\phi$ on $\mathbf{n}$. Hence

$$V_B - V_A \cong l_{AB} \mid \nabla\phi \mid_0 \cos\alpha, \tag{7}$$

$$I_{AB} \cong -\sigma_0 r_{12} \mid \nabla\phi \mid_0 \cos\alpha. \tag{8}$$

The value of the resistor connecting A and B is

$$R_{AB} = \frac{V_A - V_B}{I_{AB}} = \frac{l_{AB}}{\sigma_0 r_{12}}. \tag{9}$$

Hence R_{AB} depends only on the physical properties of the material and the manner in which the region is subdivided. If the segment 1-2 were not perpendicular to AB, the

value of the resistor would depend on the orientation of the field. It can also be shown by a simple geometrical argument that

$$Y_{AB} = \frac{1}{R_{AB}} = \frac{\sigma_0}{2} (\text{ctn } \beta_1 + \text{ctn } \beta_2), \tag{10}$$

where β_1 and β_2 are the interior angles of the triangles subtended by the segment AB. If both β_1 and β_2 are acute angles R_{AB} will be physically realizable.

In addition to calculating the value of R_{AB} it is necessary to decide on an area element to be associated with the inhomogeneous term, τ, of Eq. (4). If Eq. (4) is integrated over the polygon 1-2-3-4-5 surrounding point B of Fig. 4,

$$\iint_B \nabla \cdot (\sigma \nabla \phi)\, dS + \iint_B \tau\, dS = 0. \tag{11}$$

By Gauss' integral theorem:

$$\iint_B \nabla \cdot (\sigma \nabla \phi)\, dS = \oint_B \sigma(\nabla \phi \cdot \mathbf{n})\, dr. \tag{12}$$

Hence

$$\oint_B \sigma(\nabla \phi \cdot \mathbf{n})\, dr + \iint_B \tau\, dS = 0. \tag{13}$$

If the surface integral in Eq. (13) is replaced by the value of τ measured at B multiplied by the area of the dotted polygon and the line integral is replaced by network currents from Eq. (6):

$$\sum_p I_{pB} + \tau_B A_B = 0, \tag{14}$$

where I_{pB} is the current flowing into node B from the p-th adjacent node. Eq. (14) is Kirchoff's law for the sum of the currents entering a node. It shows that the appropriate area for calculating the current to be inserted into node B is the area interior to the dotted polygon surrounding B.

By substituting Eq. (9) into Eq. (14) the generalized difference equation for node B is obtained:

$$\sum_p \sigma_0 \left(\frac{r_{Bp}}{l_{Bp}}\right)(V_p - V_B) + \tau_B A_B = 0, \tag{15}$$

where l_{Bp} is the distance between node B and node p and r_{Bp} is the length of the segment that is common to the polygons surrounding node B and node p.

The method that has been described will work for any network configuration in which the perpendicular bisectors of the branches meet at a point. Thus besides for triangles, the method will work for rectangles, regular hexagons and isosceles trapezoids. Later it will be shown that the perpendiculars to the sides need not bisect the sides, so long as they meet at common points.

From the method of derivation given here nothing can be inferred as to the accuracy of the solutions of Eq. (15), except that for a region of uniform conductivity with a constant, arbitrarily oriented potential gradient, the solutions will yield correct answers to the field problem. The magnitude of the errors will be investigated in a later section.

Applications of the asymmetrical network. The manner in which the asymmetrical network can be applied to the problem of a curved boundary is illustrated in Fig. 5. A certain number of points are placed on the boundary and lines joining boundary points are considered in the same manner as lines joining interior points, except that the conductivity of material outside the boundary is set equal to zero. If the outward normal gradient of the field, $\partial\phi/\partial n$, is specified at the boundary, an additional current equal to $\partial\phi/\partial n$ multiplied by the conductivity and the length of boundary associated with each

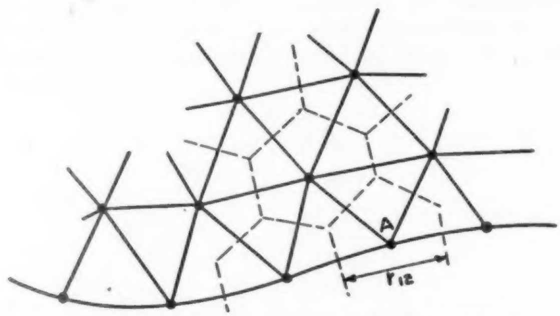

FIG. 5. Asymetrical network near a curved boundary.

node is fed into each boundary node. For point A of Fig. 5 for example this additional current is $\sigma_A[\partial\phi/\partial n]_A\ r_{12}$. This current is equal to the total flux crossing the boundary along the segment r_{12}. Hence for the asymmetrical network, boundary points are treated in almost the same manner as interior points.

The manner in which the principles of the asymmetrical network can be used to change cell size in a network of squares is illustrated in Figs. 6a and 6b. The numbers

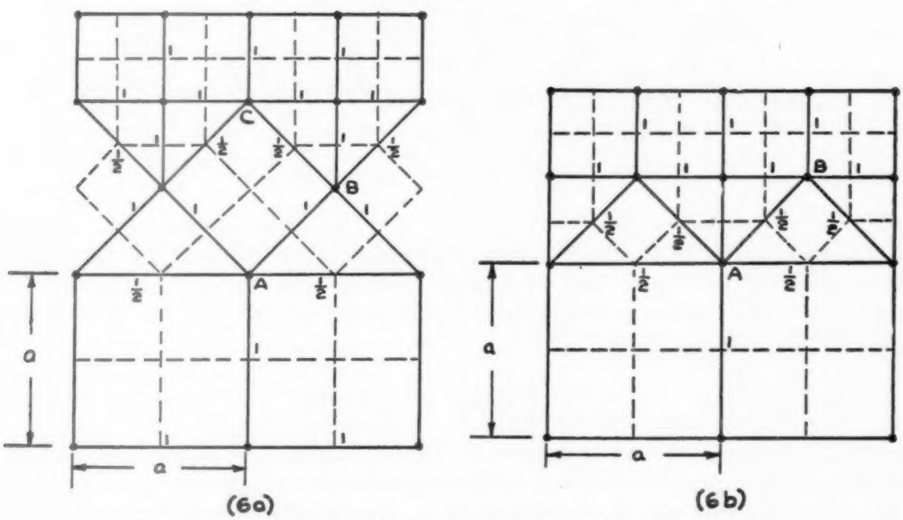

FIG. 6. Two ways in which to double cell size in a network of squares.

beside each branch of the network are the ratios r_{Bp}/l_{Bp} from Eq. (15). For example the branch connecting nodes B and C of Fig. 6a has a length $a/\sqrt{2}$ while the length of the common boundary is $a/2\sqrt{2}$ giving to r_{BC}/l_{BC} the value 1/2. It will be noted that the only values of this ratio occurring in these figures are 1/2 and 1. In contrast with the network of Fig. 1, the networks of Fig. 6 can be physically constructed and used for an analog computer solution. Another advantage that should not be underestimated is that the current in each element of the networks has a real physical significance. It represents the total normal flux crossing a known line segment.

An example of the application of the asymmetrical network to a complete problem is illustrated in Fig. 7. The problem concerns the calculation of the resonant frequencies

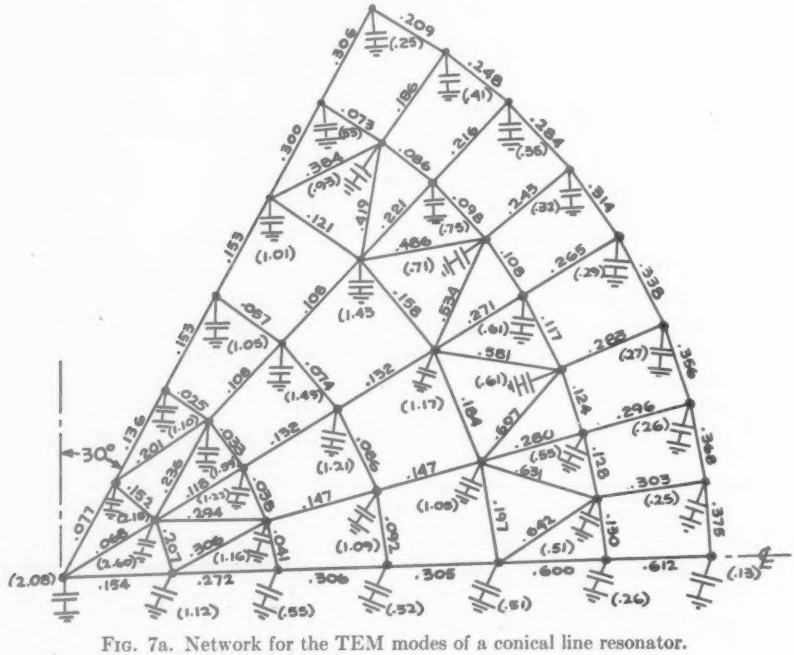

FIG. 7a. Network for the TEM modes of a conical line resonator.

and field patterns of the so-called conical-line cavity resonator, a cross-section of which is illustrated in Fig. 7c. For TEM modes (modes in which electric field lines lie in planes passing through the vertical axis and magnetic field lines are concentric circles surrounding the axis) the equation governing the variation of the magnetic field in a plane passing through the vertical axis is

$$\nabla\cdot\left(\frac{1}{\rho}\,\nabla H_3\right) + \frac{\lambda^2}{\rho}\,H_3 = 0, \tag{16}$$

where ρ is the perpendicular distance to the vertical axis, H_3 is the covarient component of magnetic field intensity and λ^2 is an eigenvalue related to the frequency of oscillation. The physical component H_ϕ is equal to H_3/ρ. The boundary condition on H_3 is that $\partial H_3/\partial n = 0$ along the walls of the cavity. The cavity is assumed to have the shape of a

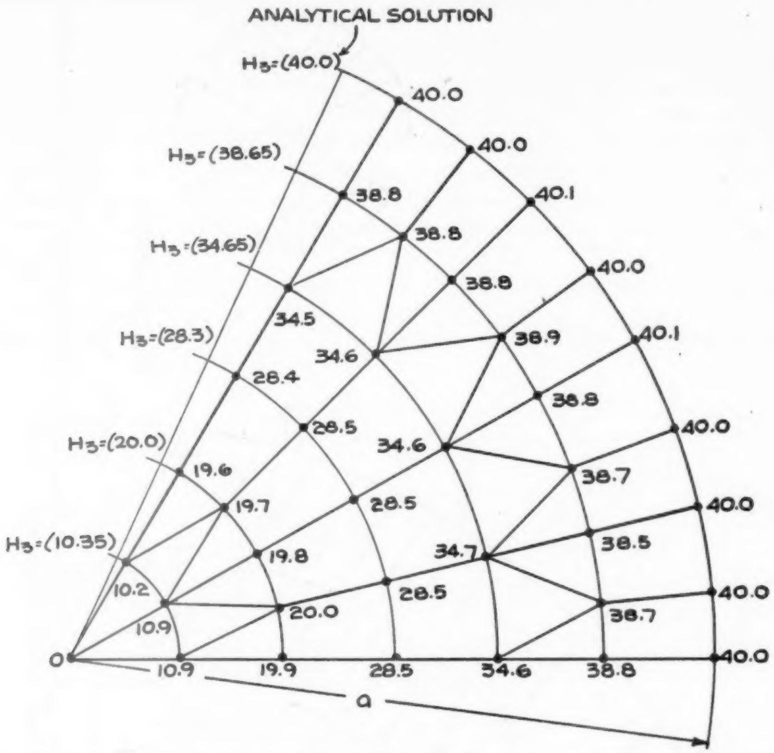

FIG. 7b. Computer solution of the lowest mode of a conical line resonator. Measured wavelength = 3.984a. Correct wavelength = 4a.

sphere where 30° conical dimples at the poles. By comparison with Eq. (4) we see that $\sigma = 1/\rho$ and $\tau = \lambda^2 H_3/\rho$.

Solutions for this problem were obtained by the electrical analogy method. An electrical circuit together with numerical values of the network elements are shown in Fig 7a. Since the coefficients of the second term in Eq. (16) is inherently positive and varies with the frequency of oscillation, a variable "negative" resistance is required for its realization. This difficulty is avoided by using inductors for the elements between nodes and capacitors for the negative elements. If the network is resonating with a frequency ω, the equation for the sum of currents at any node, B, is:

$$\sum_p \frac{1}{i\omega L_{Bp}} (V_p - V_B) - i\omega C_B V_B = 0. \tag{17}$$

Upon multiplying this equation by $i\omega$ and comparing the result with equation (15), using the values of σ and τ appropriate to this problem, it becomes apparent that:

$$L_{Bp} = \rho \frac{l_{Bp}}{r_{Bp}}, \qquad C_B = \rho A_B\,, \qquad \omega^2 = \lambda^2. \tag{18}$$

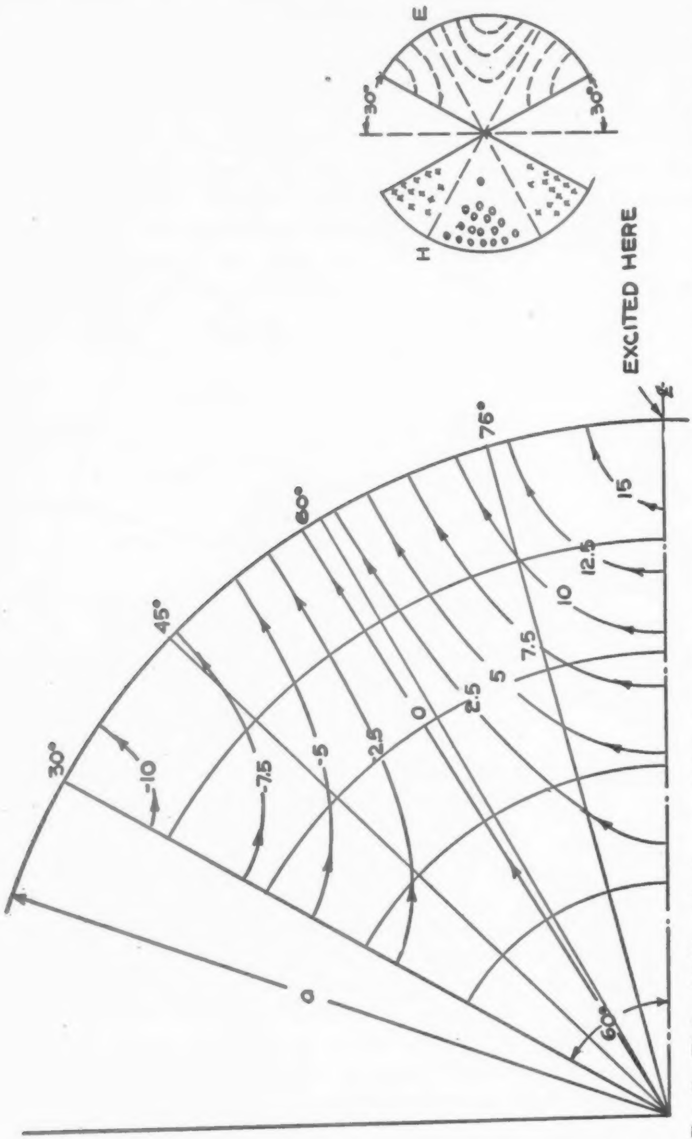

Fig. 7c. Higher TEM mode of a conical line resonator. Contours are lines of constant H_3, which are parallel to the electrical field. Measured wavelength = 1.457a.

The resonant frequencies and the corresponding eigenvalues were obtained experimentally.

The eigenfunction for the lowest mode of the conical-line resonator has the simple form

$$H_3 = \sin \frac{\pi r}{2a}, \tag{19}$$

where r is the distance from the center and a is the radius of the sphere. This solution is compared with values measured on the network in Fig. 7b. Lines of constant H_3, which are parallel to the electric field, are plotted in Fig. 7c for a higher TEM mode. In Fig. 7a it is seen that the principles of the asymmetrical network have been used to fit the location of nodes to the natural boundaries of the cavity resonator and also to effect changes in cell size so as to keep the cell area nearly constant in going from the center to the outer wall. The solutions were obtained on the California Institute of Technology Electric Analog Computer [8].

The use of Taylor's series expansions. A network for changing cell size within a network of squares, which has not been constructed according to the principles of the preceding section, is shown in Fig. 8. This network appeared without much explanation

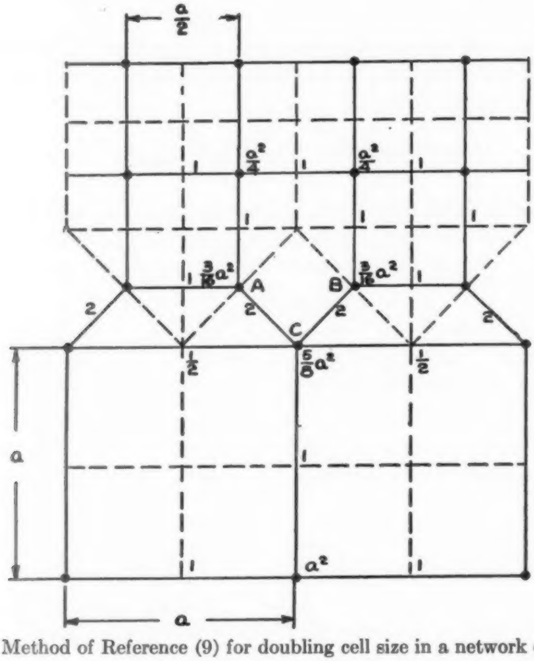

FIG. 8. Method of Reference (9) for doubling cell size in a network of squares.

in a paper by Spangenberg, Walters, and Schott [9]. The numbers beside each branch are coefficients corresponding to r_{B_p}/l_{B_p} in Eq. (15) and the numbers beside each node are the areas to be associated with each node. If this network had been constructed by the methods of the preceding section there would be, for example, a branch connecting nodes A and B. The values of the branch coefficients shown in the figure can be obtained if perpendiculars are drawn not through the midpoints of the branches but in the manner shown in Fig. 8. (These lines were not drawn in the paper quoted above.) This is permissible since, in the derivation leading to Eq. (15), the assumption that the perpendiculars to the branch segments were drawn through the *midpoints* of these segments was not essential. If the area of the dotted polygons surrounding each node are now computed, the results do not agree with the values given in the figure. For example the

area of the polygon surrounding node C is $3/4\ a^2$ rather than $5/8\ a^2$. The question is thus raised as to which of these values is better. This and other questions can be investigated by means of an error analysis using Taylor's series expansions.

The following investigation is limited to the case where the coefficients of Eq. (4) are constants, and for simplicity these coefficients will be set equal to unity. The results of the analysis are valid for any values of τ and σ. With

$$\nabla^2\phi + 1 = 0, \tag{20}$$

the corresponding generalized difference equation for the potential ϕ_0 of any node of the network is

$$\sum_p Y_{p0}(\phi_p - \phi_0) + A_0 = 0, \tag{21}$$

where ϕ_p is the potential at a neighboring node and A_0 is an element of area to be associated with ϕ_0. By a comparison of these equations we see that a measure of the error introduced by replacing Eq. (20) by Eq. (21) is

$$\epsilon_0 = A_0(\nabla^2\phi)_0 - \sum_p Y_{p0}(\phi_p - \phi_0). \tag{22}$$

If the point where ϕ_0 is defined is taken as the origin of a cartesian system of coordinates, the potential at any neighboring point can be obtained by a Taylor's Series expansion. The summation in Eq. (22) can be expressed in terms of such a series as

$$\sum_p Y_{p0}[\phi_p - \phi_0] = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_p Y_{p0}\left(x_p \frac{\partial}{\partial x} + y_p \frac{\partial}{\partial y}\right)_0^n \phi. \tag{23}$$

The coefficients of the terms of this series are arranged in tabular form below:

$$\begin{array}{ll}
\textit{term} & \textit{coefficient} \\
\left(\dfrac{\partial\phi}{\partial x}\right)_0 & C_1 = \sum_p Y_{p0}x_p \\
\left(\dfrac{\partial\phi}{\partial y}\right)_0 & C_2 = \sum_p Y_{p0}y_p \\
\left(\dfrac{\partial^2\phi}{\partial x^2}\right)_0 & C_3 = \dfrac{1}{2}\sum_p Y_{p0}x_p^2 \\
\left(\dfrac{\partial^2\phi}{\partial x\,\partial y}\right)_0 & C_4 = \sum_p Y_{p0}x_py_p \\
\left(\dfrac{\partial^2\phi}{\partial y^2}\right)_0 & C_5 = \dfrac{1}{2}\sum_p Y_{p0}y_p^2 \\
\text{etc.} & \text{etc.}
\end{array} \tag{24}$$

In this table all coefficients except C_3 and C_5 should vanish independently if the finite difference approximation is to be correct. The vanishing of the first two coefficients is equivalent to the statement, pertaining to an analogous problem in statics, that the center of gravity of loads, Y_{p0}, concentrated at the surrounding node points should be at the origin. It will be demonstrated that this will be the case if the Y_{p0}'s are calculated

by a simple geometrical method to be described. In Fig. 9 a closed polygon is shown whose sides are each drawn perpendicular to line segments radiating from a common point. A concentrated load is placed at the end of each line segment equal to r_p/l_p where l_p is the length of the segment and r_p is the length of the side of the polygon perpendicular

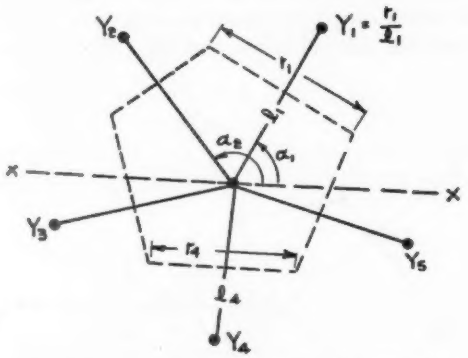

FIG. 9. Construction to prove that coefficients C_1 and C_2 vanish.

to l_p. The moment of the loads about any line, $X - X$, drawn through the common point making an angle α_p with each radiating line is

$$\text{Moment} = \sum_p \frac{r_p}{l_p} l_p \sin \alpha_p = \sum_p r_p \sin \alpha_p \tag{25}$$

The last expression is the sum of the projections of the sides of the polygon on the line $X - X$, which vanishes because the polygon is closed. Note that in this proof the polygon need not be drawn through the midpoints of the radiating lines.

In general the coefficients C_3 and C_5 of Eq. (24) will not be equal. If they are not, these terms should be combined to give a Laplacian and a "hyperbolic" operator, i.e.,

$$C_3\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + C_5\left(\frac{\partial^2\phi}{\partial y^2}\right)_0 = \frac{C_3 + C_5}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)_0 + \frac{C_3 - C_5}{2}\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right)_0. \tag{26}$$

The second term on the right can be put into the form, $(\partial^2\phi/\partial\xi\,\partial\eta)_0$, by a coordinate rotation through 45° while the first term is invariant in a coordinate rotation. From Eq. (22), the coefficient, $(C_3 + C_5)/2$, should be equal to A_0, and this provides a means for choosing the appropriate area for each cell.

$$A_0 = \frac{C_3 + C_5}{2} = \frac{1}{4}\sum_p Y_{p0}(x_p^2 + y_p^2). \tag{27}$$

If Y_{p0} is again considered as a load acting at x_p, y_p the appropriate area should be 1/4 of the sum of the polar moments of inertia of the loads.

If the value of Y_{p0} is substituted into Eq. (27), then,

$$A_0 = \frac{1}{4}\sum_p r_p l_p \tag{28}$$

since

$$l_p^2 = x_p^2 + y_p^2 .$$

If the dotted polygon of Fig. 9 passes through the midpoints of the radiating line segments, the area of the polygon satisfies Eq. (28). In the general case it seems desirable to construct boundaries which clearly delineate the area to be associated with each node. A construction which satisfies this condition and Eq. (28) is shown in Fig. 10. The ap-

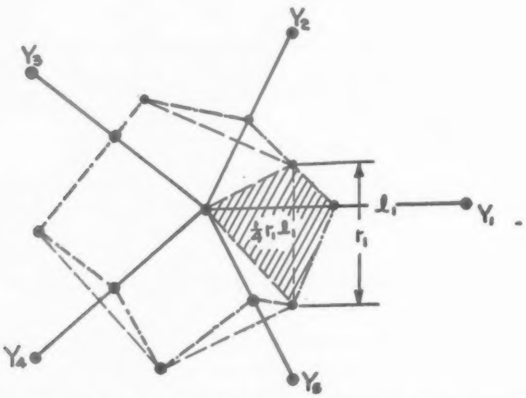

FIG. 10 Construction for obtaining the region appropriate to each nodal point.

propriate area is the area enclosed by joining the midpoints of the radiating line segments to the points of intersection of the perpendiculars to the segments. This procedure gives the same values of area coefficients for the network of Fig. 8 as those computed by by Spangenberg, Walters, and Schott [9].

Referring to Fig. 4, it can be seen that a network capable of solving the potential problem can be constructed in which the branches are the lines drawn perpendicular to the branches of the original network and in which the nodes are the vertices of the "dotted" polygons. For this new network the roles of r_p and l_p are interchanged, and this new network may be called the "geometrical dual" of the original one. Any network constructed according to the rules given in this paper has a geometrical dual. It will be observed that Fig. 8 is very nearly the geometrical dual of Fig. 6b.

The values of Y_{p0} and the area coefficient A_0 have been chosen so as to satisfy three conditions at every node. Since on the average there will be about five or six branches radiating from each node it seems that it should be possible to choose the values of the Y_{p0}'s to satisfy about three more conditions. However each Y_{p0} occurs in the equations for two nodes so that on the average for each node there are only about three independent coefficients. The non-vanishing terms of Eq. (23) will be simply regarded as errors which can, if desired, be incorporated as correction terms in the late stage of the calculation. ϵ_0 in Eq. (22) can be estimated from a trial solution and added to A_0 in Eq. (21). This follows the general procedure used in relaxation calculations [3].

The error series can be easily calculated from Eqs. (22) and (23) for each node of any given configuration. For a network of squares, with branch length a, the terms of lowest

order in the error series are

$$\epsilon_0 = -\frac{a^4}{12}\left(\frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial y^4}\right)_0. \tag{29}$$

For a network of equilateral triangles with branch length a,

$$\epsilon_0 = -\frac{a^4}{16}(\nabla^4\phi)_0 \,. \tag{30}$$

For nodes without double symmetry in their branch patterns terms of lower order will occur.

The following table gives the errors at each type of asymmetrical node in Figs. 6a, 6b and 8. Terms of the fourth order and higher are not included.

	ϵ_0
Fig. 6a	
Node A	$+\frac{a^3}{8}\left(\frac{\partial^3}{\partial y^3} - \frac{\partial^3}{\partial x^2\,\partial y}\right)_A\phi$
Node B	$\left[\frac{a^2}{16}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) + \frac{a^3}{16}\left(\frac{\partial^3}{\partial x^2\,\partial y}\right)\right]_B\phi$
Node C	$\left[-\frac{a^2}{16}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) + \frac{a^3}{16}\left(\frac{\partial^3}{\partial x^2\,\partial y}\right)\right]_C\phi$
Fig. 6b	
Node A	$\left[\frac{a^2}{16}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) - \frac{a^3}{16}\left(\frac{\partial^3}{\partial x^2\,\partial y} - 2\,\frac{\partial^3}{\partial y^3}\right)\right]_A\phi$
Node B	$\left[-\frac{a^2}{16}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) + \frac{a^3}{16}\left(\frac{\partial^3}{\partial x^2\,\partial y}\right)\right]_B\phi$
Fig. 8	
Node A	$\left[\frac{a^2}{8}\,\frac{\partial^2}{\partial x\,\partial y} + \frac{a^3}{64}\left(\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial x^2\,\partial y} - \frac{\partial^3}{\partial x\,\partial y^2} - \frac{\partial^3}{\partial y^3}\right)\right]_A\phi$
Node B	$\left[-\frac{a^2}{8}\,\frac{\partial^2}{\partial x\,\partial y} + \frac{a^3}{64}\left(-\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial x^2\,\partial y} + \frac{\partial^3}{\partial x\,\partial y^2} - \frac{\partial^3}{\partial y^3}\right)\right]_B\phi$
Node C	$\left[-\frac{a^2}{32}\left(\frac{\partial^3}{\partial x^2\,\partial y} - 5\,\frac{\partial^3}{\partial y^3}\right)\right]_C\phi$

For each figure, the sum of the ϵ_0's in a vertical strip of width a is equal to $(a^3/8)(d^3\phi/dy^3)$. This indicates that the other terms contribute to a merely local distortion of the field pattern and that on a large network this distortion will be negligible. It is interesting that for a one dimensional network in which a two to one change in cell size is made, the leading term in the error series at the point where the change is made is just $(a^3/8)$ $(d^3\phi/dy^3)$.

Conclusion. A method has been described for constructing an asymmetrical finite difference network that can be used in the solution of second order boundary value problems. The coefficients of the difference equations that govern the network can be found by simple geometrical measurements. The asymmetrical network has the

advantages that it provides a simple solution to the problem of fitting a gridwork to a curved boundary, and that it provides a means of changing cell size in such a way that the network is "realizable" by means of physical electrical elements. A clear interpretation has been given to the currents which flow along the branches of the network.

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LEGENDRE FUNCTIONS OF FRACTIONAL ORDER*

BY

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Introduction. In the theory of the propagation of spherical waves in free space the angular wave functions are Legendre polynomials, $P_n(\cos\theta)$, or associated Legendre polynomials, $P_n^m(\cos\theta)$, where n and m are restricted to integral values. These functions are polynomials in $\cos\theta$, their properties have been widely studied, numerical values have been tabulated, and in general they may be regarded as known functions.

In more recent years, however, Legendre functions of non-integral order, which we shall denote by $P_\nu(\cos\theta)$, have also occurred in physical problems. Thus, for wave propagation inside a circular horn of given angle, the boundary conditions introduce a characteristic equation which is actually an equation in the parameter ν. It has been customary to simplify the problem by choosing horn angles corresponding to integral values of ν, but a complete solution should include a study of the behavior of $P_\nu(\cos\theta)$ as a function of ν.

Similarly, in the mode theory of antennas developed by Schelkunoff the appropriate angular wave functions in the antenna region are Legendre functions of order $n + 120/K$, where n is an integer and K is the characteristic impedance of the biconical antenna to the principal wave. For thin cones K is large and the order of the Legendre functions is nearly, but not quite, integral. Further, when the cone angle is large, ν may have quite general real values.

Another application has appeared early this year, when P. Grivet† used Legendre functions of fractional order in the approximate solution of an electron lens problem, with particular emphasis on small values of ν.

Thus it appears that the properties of Legendre functions of non-integral order are of quite general interest, and it may be worth while to put on record some formulas that were developed a few years ago in connection with Schelkunoff's antenna theory. At that time the formulas were used to compute values of $P_\nu(\cos\theta)$, $0 \leqq \theta < \pi$, for values of ν between 0 and 2 at intervals of 0.1, and curves based on these computations have already been published.** Those curves show $P_\nu(\cos\theta)$ as a function of θ for the fractional values of ν; in this memorandum we include a table of numerical values (Appendix, Table I), and also a new set of curves (Figure 1) showing $P_\nu(\cos\theta)$ as a function of ν for values of θ between 0° and 175°. We have confined our computations to real values of ν, but it might be worth noting that the approximate formulas, and in particular the fundamental series expansions (3) and (17), are also valid for complex values of ν, in all regions in which they converge.

The function $P_\nu(\cos\theta)$ has a logarithmic singularity at $\theta = \pi$ for all non-integral values of ν; and it may be expressed in closed form at $\theta = \pi/2$ for all values of ν; hence

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†P. Grivet and M. Bernard, *Théorie de la lentille électrostatique constituée par deux cylindres coaxiaux*, Ann. de Radioélect., **6**, 1-9 (1952); P. Grivet, *Un nouveau modèle mathématique de lentille électronique*, Jour. de Phys. et le Rad., **13**, 1A-9A (1952).

**S. A. Schelkunoff, *Applied mathematics for engineers and scientists*, D. Van Nostrand, New York, 1948, pp. 423-424.

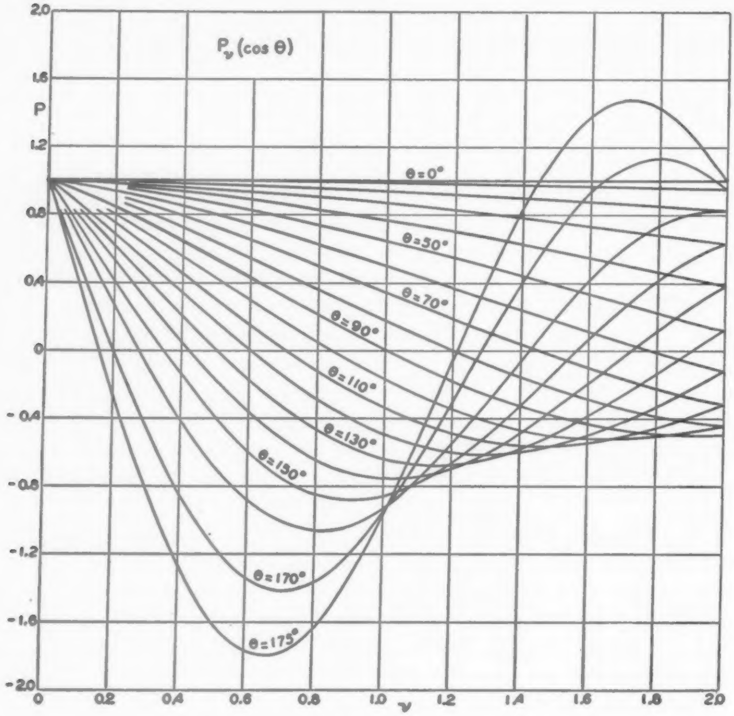

Fig. 1.

it is convenient to consider separately appropriate expansions in the neighborhoods of $\theta = 0, \pi/2, \pi$ respectively.

It may also be pointed out that for nearly integral values of ν, say $\nu = n + \delta$, it is sufficient to consider the case of $n = 0$ and small (positive or negative) values of δ. Then the recurrence formulas

$$P_{1+\delta}(\cos\theta) = \frac{1+2\delta}{1+\delta}\cos\theta P_\delta(\cos\theta) - \frac{\delta}{1+\delta}P_{-\delta}(\cos\theta),$$

$$P_{n+\delta}(\cos\theta) = \frac{2n-1+2\delta}{n+\delta}\cos\theta P_{n-1+\delta}(\cos\theta) - \frac{n-1+\delta}{n+\delta}P_{n-2+\delta}(\cos\theta), \tag{1}$$

may be used to obtain the required values for all values of n.

1. Small values of θ. When θ is not too large, the usual series expansion,*

$$P_\nu(\cos\theta) = \sum_{s=0}^{\infty}\frac{(-)^s(\nu+s)!}{(\nu-s)!s!s!}\sin^{2s}\frac{1}{2}\theta, \tag{2}$$

converges quite rapidly and tables of the factorial function are available. This series, however, does not exhibit the analytic nature of $P_\nu(\cos\theta)$ as a function of ν. Hence,

*S. A. Schelkunoff, *loc. cit.*, p. 420.

for small values of ν, we developed a series expansion in powers of ν,

$$P_\nu(\cos\theta) = 1 + \sum_{n=1}^{\infty} a_n \nu^n, \tag{3}$$

where the a's are functions of θ which can be computed from a set of recurrence relations. If we write $z = \sin^2(\theta/2)$ we can express the a's in the following form:

$$a_{2n+1} = \frac{(-)^{n+1}}{n!} \sum_{s=n+1}^{\infty} k_{n,s} \frac{z^s}{s}, \qquad a_{2n+2} = \frac{(-)^{n+1}}{n!} \sum_{s=n+1}^{\infty} k_{n,s} \frac{z^s}{s^2}, \tag{4}$$

where

$$\begin{aligned} k_{0,s} &= 1, \qquad s = 1, 2, \cdots \\ k_{n,s} &= 0, \qquad s \leqq n \\ k_{n,n+1} &= \frac{1}{n!}, \end{aligned} \tag{5}$$

$$k_{n,s+1} = k_{n,s} + \frac{n}{s^2} k_{n-1,s}\,, \qquad s = n+1, n+2, \cdots$$

It can be seen that the values k can be tabulated very rapidly, and then multiplied by the appropriate factors involving z to obtain the values a.

In particular

$$\begin{aligned} a_1 &= -\sum_{s=1}^{\infty} \frac{z^s}{s} = \log(1 - z) = 2 \log \cos \frac{1}{2}\theta, \\ a_2 &= -\sum_{s=1}^{\infty} \frac{z^s}{s^2}, \\ a_3 &= \sum_{s=2}^{\infty} \sigma_{2,s} \frac{z^s}{s}, \qquad a_4 = \sum_{s=2}^{\infty} \sigma_{2,s} \frac{z^s}{s^2}, \qquad \sigma_{2,3} = \sum_{r=1}^{s-1} \frac{1}{r^2}. \end{aligned} \tag{6}$$

When $\theta \leqq \pi/2$ we have $z \leqq 1/2$ and the series (4) converge quite rapidly, while the successive a's become smaller. The series (3) is valid for either positive or negative values of ν, and can be used very conveniently to compute $P_\nu(\cos\theta)$ for values of ν such that $|\nu| < 1/2$. Then the recurrence relations (1) may be used for larger values of ν, using also the general relation $P_{-\nu-1} = P_\nu$.

At $\nu = \pm 1/2$ the convergence is rather slow, except for small values of θ, but the Legendre functions can be expressed in terms of elliptic integrals,

$$\begin{aligned} P_{1/2}(\cos\theta) &= \frac{2}{\pi}\left[2E\left(\sin\frac{1}{2}\theta\right) - K\left(\sin\frac{1}{2}\theta\right)\right], \\ P_{-1/2}(\cos\theta) &= \frac{2}{\pi} K\left(\sin\frac{1}{2}\theta\right), \end{aligned} \tag{7}$$

where K and E are the complete elliptic integrals of the first and second kind, respectively. These functions have been frequently tabulated, and may be regarded as known. The

recurrence formulas enable us to express the Legendre functions of order $\nu = n + 1/2$ in terms of K and E.

We shall see later that, for small values of ν, the first approximation

$$P_\nu(\cos\theta) = 1 + 2\nu \log \cos \frac{1}{2}\theta \tag{8}$$

is good throughout the range $0 \leqq \theta < \pi$.

2. The neighborhood of $\theta = \pi/2$. When $\theta = \pi/2$ the value of the Legendre function is

$$P_\nu\left(\cos\frac{1}{2}\pi\right) = \frac{\cos\frac{1}{2}\nu\pi(\frac{1}{2}\nu - \frac{1}{2})!}{\sqrt{\pi}\,(\frac{1}{2}\nu)!} \tag{9}$$

for all values of ν. In the neighborhood of $\pi/2$ we write $\theta = \pi/2 - \alpha$ and a series expansion which converges rapidly for small values of α is

$$\begin{aligned} P_\nu\left[\cos\left(\frac{1}{2}\pi - \alpha\right)\right] = P_\nu(\sin\alpha) &= \sum_{r=0}^{\infty} \frac{(\frac{1}{2}\nu + \frac{1}{2}r - \frac{1}{2})!}{\sqrt{\pi}\,(\frac{1}{2}\nu - \frac{1}{2}r)!} \cos\frac{\nu - r}{2}\pi \frac{(2\sin\alpha)^r}{r!} \\ &= \cos\frac{1}{2}\nu\pi \frac{(\frac{1}{2}\nu - \frac{1}{2})!}{\sqrt{\pi}\,(\frac{1}{2}\nu)!} F\left(\frac{1}{2}\nu + \frac{1}{2}, -\frac{1}{2}\nu; \frac{1}{2}; \sin^2\alpha\right) \\ &\quad + 2\sin\alpha \sin\frac{1}{2}\nu\pi \frac{(\frac{1}{2}\nu)!}{\sqrt{\pi}\,(\frac{1}{2}\nu - \frac{1}{2})!} F\left(\frac{1}{2}\nu + 1, -\frac{1}{2}\nu + \frac{1}{2}; \frac{3}{2}; \sin^2\alpha\right). \end{aligned} \tag{10}$$

When we consider $P_\nu(\cos\pi/2)$ as a function of ν, it can be shown that the first few terms in the expansion of the function (9) in powers of ν are

$$\begin{aligned} P_\nu\left(\cos\frac{\pi}{2}\right) &= 1 - \nu\log 2 - \frac{\nu^2}{2!}\left[\frac{1}{2}\pi^2 - (\log 2)^2\right] \\ &\quad + \frac{\nu^3}{3!}\left[\frac{\pi^2}{2}\log 2 - (\log 2)^3 - \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}\right] + \cdots . \end{aligned} \tag{11}$$

These terms, however, check with those obtained in Section 1, since, at $\theta = \pi/2$,

$$2\log\cos\frac{1}{2}\theta = 2\log\cos\frac{1}{4}\pi = -\log 2;$$

the relation

$$\sum_{s=1}^{\infty} \frac{1}{2^s s^2} = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2$$

is given in Smithsonian Mathematical Tables, p. 142; and

$$\sum_{s=2}^{\infty} \frac{\sigma_{2,s}}{2^s s} = \frac{\pi^2}{12}(\log 2) - \frac{1}{2}(\log 2)^3 - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}$$

can be verified numerically. If we try to expand the series in equation (10) in powers of ν we find that when all terms have been collected we simply return to the series (3).

3. The neighborhood of $\theta = \pi$. In this region the most useful expansion for all values of ν seems to be that first obtained by E. Hille.* If we write $\theta = \pi - \varphi$ Hille's

*See E. W. Hobson, *Spherical and ellipsoidal harmonics*, University Press, Cambridge, 1931, p. 225.

formula may be written in the alternative forms

$$P_\nu[\cos(\pi - \varphi)] = P_\nu(-\cos\varphi)$$

$$= \frac{\sin \nu\pi}{\pi} \sum_{r=0}^{\infty} \frac{(-)^r(\nu + r)!}{(\nu - r)!}\left[2 \log \sin \frac{1}{2}\varphi + \psi(\nu + r) + \psi(-\nu + r - 1) - 2\psi(r)\right]\frac{z^r}{r!r!} \tag{12}$$

$$= \left[\frac{2 \sin \nu\pi}{\pi} \log \sin \frac{1}{2}\varphi + \cos \nu\pi\right] P_\nu(\cos\varphi)$$

$$+ \frac{\sin \nu\pi}{\pi} \sum_{r=0}^{\infty} \frac{(-)^r(\nu + r)!}{(\nu - r)!}[\psi(\nu + r) + \psi(\nu - r) - 2\psi(r)]\frac{z^r}{r!r!} \tag{13}$$

where $z = \sin^2 \varphi/2$ and $\psi(x)$ is the logarithmic derivative of the factorial, as used by Hobson,

$$\psi(x) = \frac{d}{dx}(\log x!).$$

When φ is small the series in equation (13) converges rapidly and only a few terms are significant. Further, the first term of the series vanishes with ν for small ν and thus the dominant terms in the expansion in powers of ν are those obtained from the first term in equation (13),

$$\begin{aligned} P_\nu(\cos\theta) = P_\nu(-\cos\varphi) &= P_\nu(\cos\varphi)\left(1 + 2\nu \log \sin \frac{1}{2}\varphi\right) \\ &\simeq 1 + 2\nu \log \cos \frac{1}{2}\theta. \end{aligned} \tag{14}$$

If we include the series terms, the expansion in powers of ν for small φ found most convenient for computation may be written

$$P_\nu(-\cos\varphi) = P_\nu(\cos\varphi)\left[\cos \nu\pi + \frac{2 \sin \nu\pi}{\pi}\left(\log \sin \frac{1}{2}\varphi + \psi(\nu) - \psi(0)\right)\right]$$

$$+ \frac{\sin \nu\pi}{\pi}(c_0 + c_1\nu + c_2\nu^2 + \cdots), \tag{15}$$

where

$$\begin{aligned} c_0 &= -a_1\,, \\ c_1 &= -2a_2\,, \\ c_2 &= -a_3 + 2\sum_{s=1}^{\infty} \frac{z^s}{s^3}, \\ c_4 &= -2a_4 - 2\sum_{s=2}^{\infty} \sigma_{2,s}\frac{z^s}{s}, \\ c_5 &= -a_4 - 2\sum_{s=2}^{\infty} \sigma_{2,s}\frac{z^s}{s^3} - 2\sum_{s=2}^{\infty} \sigma_{3,s}\frac{z^s}{s^2} \end{aligned} \tag{16}$$

and where the values a are the constants of $P_\nu(\cos\varphi)$, $z = \sin^2\varphi/2$, and

$$\sigma_{n,s} = \sum_{r=1}^{s-1} \frac{1}{r^n}.$$

When φ is not too small it is also possible to use the series expansion

$$P_\nu(-\cos\varphi) = 1 + \sum_{n=1}^{\infty} b_n \nu^n, \tag{17}$$

with

$$\begin{aligned}
b_1 &= 2\log\sin\frac{1}{2}\varphi,\\
b_2 &= a_1 b_1 - a_2 - \frac{\pi^2}{6},\\
b_3 &= b_1\left(a_2 - \frac{\pi^2}{6}\right) + 2\sum_{s=1}^{\infty}\frac{z^s}{s^3} - 2\sum_{s=1}^{\infty}\frac{1}{s^3},\\
b_4 &= b_1\left(a_3 - \frac{\pi^2}{6}a_1\right) - 2a_1\sum_{s=1}^{\infty}\frac{1}{s^3} + \frac{\pi^2}{6}a_2 - a_4 + \frac{\pi^4}{120} - 2\sum_{s=2}^{\infty}\sigma_{3,s}\frac{z^s}{s}.
\end{aligned} \tag{18}$$

The coefficients are of course more complicated than those of equation (3), but if the a's have already been computed the remaining terms may be evaluated without too much labor.

4. Zeros of $P_\nu(\cos\theta)$. In Grivet's electron lens theory certain focal distances are determined from the roots θ_0 of $P_\nu(\cos\theta) = 0$ and from the values of $dP_\nu/d\theta$ at $\theta = \theta_0$. From the values of Table I the curve of Figure 2 has been drawn, showing the roots

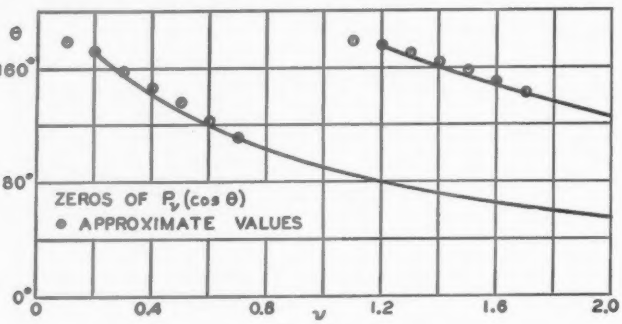

FIG. 2.

of P_ν for values of ν between 0 and 2. However, approximate values of the roots can be found from the approximate formulas of the preceding sections, and it is surprising that the simple formula (8) gives values very near the true values except in the neighborhood of $\nu = 1/2$. At this point the root can be obtained from elliptic integral tables. For smaller values of ν we find from (8)

$$\cos\frac{1}{2}\theta = \exp\left(-\frac{1}{2\nu}\right), \tag{19}$$

and for values of ν between 0.5 and 1.5 we combine equation (8) with the recurrence relation (1) to derive the equation

$$\log \cos \frac{1}{2}\,\theta = -\frac{1}{2\delta}\frac{(1 + 2\delta)\cos\theta - \delta}{(1 + 2\delta)\cos\theta + \delta}, \qquad \nu = 1 + \delta. \tag{20}$$

Similar equations can be obtained for larger values of ν by repeated use of the recurrence formula, and it is always possible to obtain an equation for the root which expresses $\log\cos\theta/2$ as the ratio of two polynomials in $\cos\theta$.

When ν is small the root θ_0 is near π, and a somewhat more accurate formula is obtained from equation (15):

$$\log \sin \frac{1}{2}\,\varphi = \frac{\log \cos \frac{1}{2}\varphi}{1 + 2\nu \log \cos \frac{1}{2}\varphi} - \frac{\pi}{2}\cot \nu\pi - \psi(\nu) + \psi(0), \tag{21}$$

where $\theta = \pi - \varphi$. Similarly for $\nu = 1 + \delta$, where δ is small, the smallest root can be found from equation (20), but there is a second root near π which is determined more accurately from the equation

$$\log \cos \frac{1}{2}\,\theta = -\frac{\pi}{2}\cot \delta\pi\left[\frac{(1 + 2\delta)\cos\theta - \delta}{(1 + 2\delta)\cos\theta + \delta}\right] - \frac{\cos\theta[\psi(\delta) - \psi(0) - \log\sin\frac{1}{2}\theta]}{(1 + 2\delta)\cos\theta + \delta}. \tag{22}$$

In Figure 2 we have indicated by circles the values of the roots obtained from equations (19) and (20) where these may be distinguished from the curve values.

For the derivative of P_ν at $\theta = \theta_0$ we can find simple formulas by differentiating the approximate formulas (3) and (15). Thus retaining the first three terms in (3) and using the approximation (19) for θ_0 we find

$$\left.\frac{dP_\nu}{d\theta}\right|_{\theta=\theta_0} = -\frac{2\nu}{\sin\theta_0}. \tag{23}$$

When ν is small, so that θ_0 is near π the asymtotic approximation is

$$\left.\frac{dP_\nu}{d\theta}\right|_{\theta=\theta_0} = -\frac{2\sin\nu\pi}{\pi\sin\theta_0}\frac{1}{1 + 2\nu\log\sin\theta_0/2}. \tag{24}$$

At $\nu = 0.5$ the derivatives of the elliptic integrals give

$$\left.\frac{d}{d\theta}P_{1/2}(\cos\theta)\right|_{\theta=\theta_0} = -\frac{K(\sin\theta_0/2)}{\pi\sin\theta_0}, \tag{25}$$

as shown by Grivet. For $\nu > 0.5$ we may combine (20) and (3) with the recurrence formula to find approximately

$$\left.\frac{d}{d\theta}P_{1+\delta}(\cos\theta)\right|_{\theta=\theta_0} = \frac{-2\delta(1 + \delta)}{\sin\theta_0[(1 + 2\delta)\cos\theta_0 + \delta]}. \tag{26}$$

Note that this equation is valid as $\delta \to 0$. For we have $P_{1+\delta} \to \cos\theta$, and the limit is approached in such a way that $\cos\theta_0 = \lim_{\delta\to0} \delta/(1 + 2\delta)$. Thus in equation (26) the limiting value is

$$\left.\frac{d}{d\theta}P_1(\cos\theta)\right|_{\theta=\theta_0} = -\frac{1}{\sin\theta_0} = -1,$$

which is correct.

APPENDIX, TABLE I

Legendre Functions of Fractional Order, $P_\nu(\cos\theta)$

θ	$\nu = .1$	.2	.3	.4	.5	.6	.7	.8	.9	1.0
10°	.999161	.998171	.997028	.995735	.994289	.992693	.990947	.989050	.987003	.984808
20°	.996635	.992665	.988095	.982927	.977168	.970822	.963895	.956393	.948322	.939693
30°	.992387	.983428	.973140	.961544	.948665	.934528	.919163	.902601	.884877	.866025
40°	.986362	.970362	.952059	.931521	.908821	.884042	.857275	.828616	.798169	.766044
50°	.978471	.953322	.924694	.892752	.857676	.819665	.778935	.735715	.692387	.642788
60°	.968597	.932102	.890814	.845072	.795249	.741748	.685007	.625480	.563646	.5
70°	.956571	.906416	.850092	.788227	.721505	.650659	.576469	.499745	.421314	.342020
80°	.942171	.875872	.802069	.721889	.636309	.546730	.454374	.360536	.266528	.173648
90°	.925086	.839927	.746089	.645288	.539353	.430189	.319752	.209982	.102787	0
100°	.904886	.797813	.681210	.557670	.430035	.301038	.173516	.050203	−.066312	−.173648
110°	.880955	.748422	.606031	.456324	.307261	.157754	.016269	−.116849	−.237227	−.342020
120°	.852374	.690081	.518406	.342882	.169084	.002434	−.151995	−.289652	−.406673	−.5
130°	.817704	.620144	.414869	.209624	.012012	−.170817	−.332468	−.467557	−.574395	−.642788
140°	.774511	.534092	.289416	.051166	−.170483	−.366364	−.528746	−.651698	−.734616	−.766044
150°	.718190	.423320	.130467	−.145688	−.391682	−.596024	−.749809	−.847187	−.885632	−.866025
160°	.638358	.268268	−.088558	−.411669	−.683193	−.888957	−1.019355	−1.069887	−1.041354	−.939693
170°	.501717	.005894	−.453932	−.847492	−1.150000	−1.343918	−1.420160	−1.378607	−1.227945	−.984808
175°	.365201	−.254581	−.813813	−1.272544	−1.599553	−1.774742	−1.791031	−1.652930	−1.378654	−.996444

θ	$\nu = 1.1$	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
10°	.982463	.979971	.977330	.974544	.971609	.968530	.965306	.961937	.958424	.954769
20°	.930510	.920782	.910521	.899731	.888427	.876616	.864311	.851520	.838256	.824533
30°	.846085	.825099	.803106	.780153	.756288	.731556	.706009	.679698	.652673	.625
40°	.732358	.697230	.660787	.623160	.584483	.544892	.504529	.463536	.422057	.380236
50°	.593260	.542960	.491146	.438457	.385144	.331534	.277906	+.224546	+.171992	+.119764
60°	.435048	.369302	.303277	.237487	+.172439	+.108625	+.046528	−.013395	−.070707	−.125
70°	.262707	+.184212	+.107352	+.032911	−.038356	−.105747	−.168608	−.226363	−.278485	−.324533
80°	+.033160	−.003726	−.085870	−.162209	−.231828	−.293895	−.347694	−.392684	−.428461	−.454769
90°	−.096662	−.185629	−.265506	−.335101	−.393447	−.439820	−.473745	−.495011	−.503662	−.5
100°	−.269688	−.352628	−.421019	−.473792	−.510309	−.530315	−.533992	−.521924	−.495078	−.454769
110°	−.423902	−.496128	−.542575	−.567190	−.571540	−.555254	−.520788	−.468913	−.402560	−.324533
120°	−.567487	−.607957	−.621232	−.608117	−.570350	−.510520	−.431952	−.338567	−.234713	−.125
130°	−.679240	−.680298	−.648801	−.587376	−.500235	−.392577	−.270327	−.139860	−.007413	+.119764
140°	−.756377	−.705113	−.617004	−.498494	−.357352	−.202235	−.042220	+.113682	+.260689	.380236
150°	−.792525	−.672251	−.514788	−.331564	−.135126	+.061648	+.246321	.407651	.536224	.625
160°	−.775450	−.562976	−.319350	−.063159	+.186880	.412909	.599730	.735204	.811308	.824533
170°	−.672324	−.318188	+.047601	+.395130	.696918	.930007	1.077683	1.130627	1.087486	.954769
175°	−.542651	−.058480	.413527	.833237	1.165932	1.383435	1.476543	1.433916	1.265476	989351

THE LOCATION OF THE ROOTS OF POLYNOMIAL EQUATIONS BY THE REPEATED EVALUATION OF LINEAR FORMS*

BY

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1. Introduction. The author was recently engaged in problems connected with the solution of polynomial equations with the aid of an electrolytic tank analog[1]. In connection with this work he evolved a simple and apparently novel computational system of locating the complex roots of polynomial equations. This system is particularly suitable for "punched card" and "digital electronic" computing machines, because it is essentially the evaluation of linear forms, repeated systematically. The present paper describes the principles of the system.

2. The connection between a polynomial and a two-dimensional field. Let the polynomial the zeros of which are to be located[2] be

$$G(Z) = \sum_{q=0}^{q=n} L_q Z^q = \sum_{q=0}^{q=n} (l_q^{(r)} + i l_q^{(i)}) Z^q \tag{1}$$

Let u be a scale factor and Z' an auxiliary variable

$$Z = uZ' \tag{2}$$

As Lucas[3] pointed out, a rational function $H(Z')$ which has $m > n$ first order poles at arbitrarily selected points $Z_s (s = 1, 2, \cdots, m)$ and whose zeros coincide with those of $G(uZ')$ can be derived by dividing $kG(uZ')$ by

$$F(Z') = \prod_{s=1}^{s=m} (Z' - Z'_s), \tag{3}$$

where k is a constant. $H(Z')$ can be expressed as the sum of partial fractions

$$H(Z') = k \frac{G(uZ')}{F(Z')} = \sum_{s=1}^{s=m} \frac{A_s}{Z' - Z'_s}, \tag{4}$$

where

$$A_s = a_s^{(r)} + i a_s^{(i)} = k \frac{G(uZ'_s)}{B_s(Z'_s)} \tag{5}$$

and

$$B_s(Z'_s) = \prod_{q=1 \text{ to } (s-1)}^{q=(s+1) \text{ to } m} (Z'_s - Z'_q). \tag{6}$$

A_s is the residue of $H(Z')$ at Z'_s. By eliminating in (1) to (4) the expressions Z', $G(Z)$, $F(Z')$, and $H(Z')$, we obtain two power series in Z. The comparison of the first coefficients

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*Received Nov. 7, 1952.

[1] L. Tasny-Tschiassny and A. G. Doe, *The solution of polynomial equations with the aid of the electrolytic tank*, Aust. J. Sci. Research **4**, 231-257 (1951).

[2] Capital letters stand for complex numbers, small letters for real ones.

[3] F. Lucas, Résolution des équations par l'électricité, C. R. Ac. Sci. Paris **106**, 645, 1072 (1888).

shows that

$$\left.\begin{aligned} &\sum_{s=1}^{s=m} A_s = 0, &&\text{if} \quad m \geq n + 2, \\ &\sum_{s=1}^{s=m} A_s = ku^{n-1}L_n\,, &&\text{if} \quad m = n + 1. \end{aligned}\right\} \tag{7}$$

The two-dimensional vector corresponding to the conjugate of $H(Z')$ is proportional to the field strength at Z' of a two-dimensional field produced by sources of the intensity $a_s^{(r)}$ and vortexes of the intensity $a_s^{(i)}$ positioned at the poles Z'_s. If $\sum_{s=1}^{s=m} A_s = 0$, the field is self-contained, if $\sum_{s=1}^{s=m} A_s \neq 0$, a source and a vortex of intensities given by $(-\sum_{s=1}^{s=m} A_s)$ is to be imagined at $Z' = \infty$. Since the zeros of $H(Z')$ and $G(uZ')$ coincide and the zeros of $H(Z')$ are saddle points of the potential, these saddle points determine the roots of the equation $G(uZ') = 0$. This relation has been utilized for electrolytic tank analogs[4,5].

The second statement of (4) can be employed for a purely computational exploration of the field conditions. By a systematic cut and try method one can approach the points Z'_r for which $H(Z'_r) = 0$, with any desired accuracy. In general, this method will be very cumbersome, the main reason being that the accuracy required in the computation of the terms $A_s/(Z' - Z'_s)$ is considerably greater than the accuracy obtained in the result $H(Z')$.

In the special arrangements discussed in the present paper the described computational exploration becomes very simple, because it is essentially the evaluation of linear forms

$$f = \sum a_q b_q \tag{8}$$

in which the set of quantities a_q depends on the special numerical problem in hand, and the set of quantities b_q is taken from tables compiled once and for all. Digital electronic and punched card machines are particularly suitable for this type of work, but it appears that a satisfactory efficiency can be obtained with ordinary commercial multiplying machines. When evaluating linear forms on these machines, no need arises to make a record of intermediate products, because when a certain number has appeared in the result register, a further multiplication adds or subtracts the additional product to this number. If the numbers a_q are recorded on a strip of paper in a way that in a certain position all numbers a_q can be made adjacent to the corresponding numbers b_q of the table, the corresponding multiplicands and multipliers can be directly read off without risk of errors. The tables for b_q can be arranged in a way that the same a_q-strip can be used for different b_q-sets.

3. The computation of the residues A_s for symmetrically arranged poles Z'_s. A convenient number is chosen for the scale factor u in a way that at least some of the points Z'_r for which $G(uZ'_r) = 0$, are within the circle of unit radius with the origin as centre (in the following called the "unit circle"). Let the poles Z'_s be the complex roots of the equation

$$Z'^m = 1,$$

[4]A. R. Boothroyd, E. C. Cherry, and R. Makar, *An electrolytic tank for the measurement of steady-state response, transient response, and allied properties of networks*, Proc. Instn. Elec. Engrs. **96**, 163 (1949).

[5]A. Bloch, *Solution of algebraic equations by means of an electrolytic tank*, VIIth Internat. Congr. Appl. Mech., London, Paper No. IV-28.

i.e.,

$$Z'_s = e^{i(2\pi/m)s}, \qquad (s = 1, 2, \cdots, m). \tag{9}$$

The quantity $B_s(Z'_s)$ [equation (6)] may be expressed as

$$B_s(Z'_s) = \lim_{Z' \to Z'_s} \left[\frac{Z'^m - 1}{Z' - Z'_s}\right] \tag{10}$$

which after substitution from (9) and evaluation of the indeterminate form leads to

$$B_s(Z'_s) = me^{-2\pi s i/m} \tag{11}$$

From (1), (2), and (5) we obtain for $k = m$

$$A_s = \sum_{q=0}^{q=n} (L_q u^q) e^{2\pi i s(q+1)/m}, \qquad (s = 1, 2, \cdots, m). \tag{12}$$

The real and imaginary parts of A_s are linear forms of the type of equation [8], because the trigonometrical functions appearing in them can be tabulated once and for all, if m is fixed. Equations [7] can be used as a check for the absence of errors in the computations.

In the language of electrical power engineering the quantities A_s are m-times the "symmetrical coordinates" of the quantities $(L_q u^q)$[6,7]. Previous authors[8] have expressed the different roots of a polynomial equation in terms of their symmetrical coordinates, but these investigations have no bearing on the present problem.

4. The computation of H(Z′) for symmetrically arranged poles Z'_s. Let the unit circle be divided into g equal sectors. g is either equal to m or a multiple of it.

$$g = k'm \tag{13}$$

Let the radius of the unit circle be divided into $1/p_0$ equal parts ($1/p_0$ is an integer). In Fig. 1 this subdivision is shown for $m = 8$, $g = 16$, and $1/p_0 = 4$. The subdividing radii and circles determine "grid" points by their mutual intersection. Let equation [4] be rewritten as

$$H(Z') = \sum_{v=1}^{v=g} \frac{A'_v}{Z' - Z'_v}, \tag{14}$$

where

$$Z'_v = e^{2\pi i v/g} \qquad (v = 1, 2, \cdots, g) \tag{15}$$

and

$$\left.\begin{aligned} A'_v &= a'^{(r)}_v + ia'^{(i)}_v = A_{(v/k')}, &&\text{if } v/k' \text{ is integral,} \\ A'_v &= 0 &&\text{if } v/k' \text{ is not integral.} \end{aligned}\right\} \tag{16}$$

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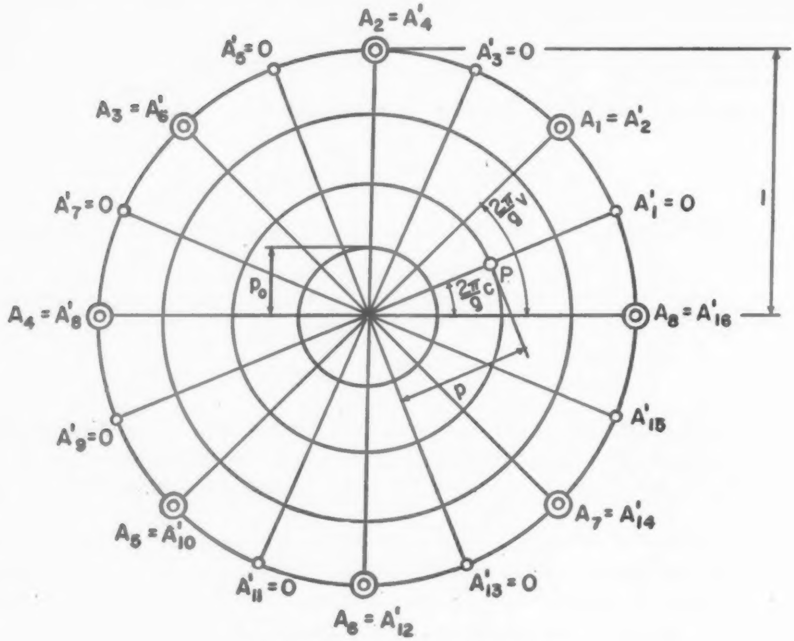
[6] C. L. Fortescue, *Method of symmetrical coordinates applied to the solution of polyphase networks*, Trans. A.I.E.E. **37**, 1315-1327 (1918).

[7] C. F. Wagner and R. D. Evans, *Symmetrical components*, First Ed., McGraw-Hill, New York and London, Ch. XVI, 328-344 (1933).

[8] C. L. Fortescue and G. Calabrese, *L'applicazione delle coordinate simmetriche alla risoluzione delle equazioni algebriche*, Atti del Congresso Internazionale dei Matematici. Bologna, p. 159 (1928).



Unit Circle and Grid Points

Let

$$Z' = pe^{i2\pi c/g} \tag{17}$$

be a grid point, so that c is an integer. By substituting (15) and (17) in (14) and introducing

$$\left.\begin{aligned} h &= v - c, \qquad &\text{if} \quad v > c \\ h &= g + v - c, \qquad &\text{if} \quad v \leq c \end{aligned}\right\} \tag{18}$$

we obtain after a few transformations

$$H(Z') = e^{-i2\pi c/g}\overline{H}(Z'), \tag{19}$$

where

$$\overline{H}(Z') = \sum_{h=1}^{h=g} \frac{A'_{h+c}}{p - e^{i2\pi h/g}}. \tag{20}$$

Let the conjugate of $\overline{H}(Z')$ be written as

$$\overline{H}^*(Z') = -(\rho + i\tau). \tag{21}$$

Then $(-\rho)$ is the radial and $(-\tau)$ the tangential component of the intensity of the field at the grid point Z'. The use of [16] and a transformation of [20] results in

$$\rho = \sum_{h=1}^{h=g} a'^{(r)}_{h+c} r_h - \sum_{h=1}^{h=g} a'^{(i)}_{h+c} t_h\,, \tag{22}$$

$$\tau = \sum_{h=1}^{h=g} a'^{(r)}_{h+c} t_h + \sum_{h=1}^{h=g} a'^{(i)}_{h+c} r_h\,, \tag{23}$$

where

$$r_h = \frac{\cos(2\pi h/g) - p}{1 + p^2 - 2p\cos(2\pi h/g)} \tag{24}$$

and

$$t_h = \frac{\sin(2\pi h/g)}{1 + p^2 - 2p\cos(2\pi h/g)}. \tag{25}$$

Equations (22) and (23) are linear forms of the type of equation (8), because the values of r_h and t_h can be tabulated once and for all. Tables 1 and 2 are examples of such tables

TABLE 1

Values of r_h for $1/p_0 = 8$ and $g = 16$.

$2\pi h/g$	$p = 0$	$p = 0'125$	$p = 0'250$	$p = 0'375$	$p = 0'500$	$p = 0'625$	$p = 0'750$	$p = 0'875$	$p = 1'000$
0°	+1′00 000	+1′14 285	+1′33 333	+1′59 998	+2′00 000	+2′66 657	+4′00 000	+8′00 000	*
22°30′	+0′92 388	+1′01 814	+1′12 210	+1′22 597	+1′29 981	+1′26 768	+0′98 421	+0′32 843	−0′50 000
45°	+0′70 711	+0′69 394	+0′64 477	+0′54 417	+0′38 150	+0′16 204	−0′08 547	−0′31 786	−0′50 000
67°30′	+0′38 268	+0′28 010	+0′15 230	+0′00 900	−0′13 527	−0′26 562	−0′37 160	−0′44 923	−0′50 000
90°	0′00 000	−0′12 308	−0′23 529	−0′32 877	−0′40 000	−0′44 944	−0′48 000	−0′49 557	−0′50 000
112°30′	−0′38 268	−0′45 684	−0′50 460	−0′53 073	−0′54 064	−0′53 916	−0′53 015	−0′51 644	−0′50 000
135°	−0′70 711	−0′69 784	−0′67 590	−0′64 760	−0′61 680	−0′58 567	−0′55 547	−0′52 683	−0′50 000
157°30′	−0′92 388	−0′84 140	−0′76 904	−0′70 840	−0′65 500	−0′60 848	−0′56 774	−0′53 183	−0′50 000
180°	−1′00 000	−0′88 889	−0′80 000	−0′72 727	−0′66 667	−0′61 538	−0′57 143	−0′53 333	−0′50 000

*The value of this limit depends on the direction of approach.

TABLE 2

Values of t_h for $1/p_0 = 8$ and $g = 16$.

$2\pi h/g$	$p = 0$	$p = 0'125$	$p = 0'250$	$p = 0'375$	$p = 0'500$	$p = 0'625$	$p = 0'750$	$p = 0'875$	$p = 1'000$
0°	0	0	0	0	0	0	0	0	*
22°30′	0′38 268	0′48 771	0′63 722	0′85 475	1′17 347	1′62 311	2′16 607	2′57 126	2′51 383
45°	0′70 711	0′84 295	0′99 741	1′15 863	1′30 252	1′39 541	1′40 903	1′33 874	1′20 713
67°30′	0′92 388	1′00 427	1′06 053	1′08 232	1′06 522	1′01 273	0′93 466	0′84 301	0′74 831
90°	1′00 000	0′98 461	0′94 118	0′87 671	0′80 000	0′71 910	0′64 000	0′56 637	0′50 000
112°30′	0′92 388	0′83 136	0′73 685	0′64 714	0′56 587	0′49 433	0′43 242	0′37 937	0′33 409
135°	0′70 711	0′59 301	0′49 935	0′42 318	0′36 131	0′31 088	0′26 954	0′23 546	0′20 711
157°30′	0′38 268	0′30 698	0′25 070	0′20 871	0′17 604	0′15 034	0′12 980	0′11 314	0′09 946
180°	0	0	0	0	0	0	0	0	0

*The value of this limit depends on the direction of approach.

for $1/p_0 = 8$ and $g = 16$. Since $r_h = r_{g-h}$ and $t_h = -t_{g-h}$, it suffices to tabulate the values between $h = 0$ and $h = g/2$.

If greater accuracy in the location of the roots is desired than determined by the mesh of radii and concentric circles, interpolation methods may be employed. $H(Z')$ is analytical and $(dH(Z'))/dZ' \neq 0$ at a root point, unless $(dG(uZ'))/dZ' = 0$ which corresponds to a multiple root. Consequently, in general, complex linear interpolation for $\overline{H}^*(Z')$ is admissible, if the values of $\overline{H}^*(Z')$ at two grid points are known. Other possibilities are repeating the described procedure for different values of u, or for equations whose roots are shifted with respect to the roots of the original equation, or for equations whose roots are powers of the roots of the original equation, etc.

After this paper had been completed a numerical method became known[9] that uses Lucas' principle to find with great accuracy "the roots of polynomial equations with exact coefficients when one has a first approximation as a starting point." Since Salzer's approach is entirely different from that in this paper, a combination of the two methods in question is not possible without a special investigation into its practicability.

5. Polynomial equations with real coefficients. It will be shown in this Section that real values of the residues A_s for a symmetrical arrangement of the poles along the unit circle can be obtained with the aid of a well-known conformal transformation, if the coefficients of the polynomial (1) are real. For real values A_s equations (22) and (23) simplify considerably.

By the relation

$$(W + i)(Z' + i) = -2 \tag{26}$$

the upper half of the Z'-plane is conformally transformed into the interior of the unit circle of the W-plane. The point $Z' = \infty$ corresponds to $W_\infty = -i$. The field configuration given by equation [4] in the Z'-plane, correlates to the field configuration given by the equation

$$H'(W) = \sum_{s=1}^{s=m} \frac{A_s}{W - W_s} - \frac{1}{W + i} \sum_{s=1}^{s=m} A_s \tag{27}$$

in the W-plane, if the poles W_s in the W-plane correspond to the poles Z'_s in the Z-plane according to [26]. Saddle points of the potential correlate in the two field maps, hence we obtain the condition

$$H'(W_r) = 0 \tag{28}$$

if W_r and Z'_r correlate. Consequently, the roots Z'_r of the equation $G(uZ'_r) = 0$ can be located by a field exploration according to Section IV carried out in the W-plane.

The steps described in this Section so far can be comprised in a single procedure. Let the uniform distribution of the points W_s including $W_\infty = -i'$ be given by

$$W_s = -ie^{i(2\pi s/n+2)}, \qquad (s = 1, 2, \cdots, n + 2). \tag{29}$$

From (26) we obtain

$$Z'_s = -\frac{\sin [2\pi s/(n + 2)]}{1 - \cos [2\pi s/(n + 2)]}, \qquad (s = 1, 2, \cdots, n + 2) \tag{30}$$

[9]Herbert E. Salzer, *On calculating the zeros of polynomials by the method of Lucas*, J. Research Nat. Bur. Stands., **49**, 133-4 (1952).

The substitution of (1) in (5) gives

$$A_s = \sum_{q=0}^{q=n} (u^q L_q)\cdot {}^s h_q, \qquad (s = 1, 2, \cdots, r + 1), \tag{31}$$

where

$${}^s h_q = k \frac{Z_s'^q}{B_s(Z_s')}, \qquad \begin{Bmatrix} (s = 1, 2, \cdots, n + 1) \\ (q = 0, 1, 2, \cdots, n) \end{Bmatrix}. \tag{32}$$

A_s is a linear form of the type of equation [8], because the quantities ${}^s h_q$ can be com-

TABLE 3
Values of ${}^s h_q$ *for* $n = 6$ $(k = 8)$.

	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$	$s = 6$	$s = 7$
$q = 0$	+0′02 512 6	−1	+4′97 487	−8	+4′97 487	−1	+0′02 512 6
$q = 1$	−0′06 066 0	+1	−2′06 066	0	+2′06 066	−1	+0′06 066 0
$q = 2$	+0′14 645	−1	+0′85 355	0	+0′85 355	−1	+0′14 645
$q = 3$	−0′35 355	+1	−0′35 355	0	+0′35 355	−1	+0′35 355
$q = 4$	+0′85 355	−1	+0′14 645	0	+0′14 645	−1	+0′85 355
$q = 5$	−2′06 066	+1	−0′06 066 0	0	+0′06 066 0	−1	+2′06 066
$q = 6$	+4′97 488	−1	+0′02 512 6	0	+0′02 512 6	−1	+4′97 488

puted and tabulated once and for all. Table 3 contains the values ${}^s h_q$ for $n = 6$. The residue at $W_\infty = W_{n+2} = -i$ is given by

$$A_{n+2} = -\sum_{s=1}^{s=n+1} A_s = -ku^{n-1}L_n . \tag{33}$$

The double statement in (33) can be used as a check for the absence of errors in the computations.

6. Methods to obtain real residues A_s for equations with complex coefficients. It may be sometimes desirable to work with real residues and to pay for this advantage with a larger number of poles. Two methods are suggested which achieve this and in which, at the same time, all roots or numbers closely associated with them are within the unit circle.

Method 1: The equation

$$\sum_{t=0}^{t=2n} k_t Z^t = \left[\sum_{q=0}^{q=n} L_q Z^q\right]\cdot\left[\sum_{h=0}^{h=n} L_h^* Z^h\right] = 0 \tag{34}$$

has real coefficients and its roots are the roots of the original equation and their conjugates. Equation (34) can be dealt with according to Section 5.

Method 2: If $G(Z)$ happens to be a polynomial of even degree in which any pair of coefficients equidistant from the two ends are complex conjugates, the residues A_s

computed from (12) turn out to be real. If the star denotes the conjugate, a subsidiary polynomial complying with this condition is

$$\sum_{t=0}^{t=2n} N_t Z'^t = \left[\sum_{q=0}^{q=n} (u^q L_q) Z'^q\right] \cdot \left[\sum_{h=0}^{h=n} (u^{n-h} L^*_{n-h}) Z'^q\right]. \tag{35}$$

The $(2n)$ zeros of the subsidiary polynomial are the n zeros of the original polynomial $G(uZ')$ and its n conjugate reciprocals.

If the values N_t are computed and substituted in (12) for $m = 2n + 2$, we obtain after some transformations

$$(-1)^s A_s = \left[\sum_{q=0}^{q=n} (u^q L_q) \cos \frac{2\pi sq}{2n+2}\right]^2 + \left[\sum_{h=0}^{h=n} (u^h L_h) \sin \frac{2\pi sq}{2n+2}\right]^2, \qquad (s = 1, 2, \cdots, 2n+2). \tag{36}$$

Equation (36) which is very closely related to linear forms of the type of equation (8) permits the direct computation of the values A_s from the coefficients of the original equation. A check for the absence of errors is the first statement of equations (7).

A METHOD OF SOLUTION OF THE EQUATIONS OF CLASSICAL GAS-DYNAMICS USING EINSTEIN'S EQUATIONS*

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Summary. It is known that Einstein's equations in general relativity provide explicit expressions for the density, pressure and velocity of a perfect gas in terms of the coefficients of the metric (the potentials) and hence in terms of the coordinates. Using orthogonal space-times, the expressions involve four potentials only between which consistency relations hold. It is shown how degeneration of the Einstein equations to Newtonian hydrodynamics provides general solutions of the equations of classical gas-dynamics for motions which may be either of constant or of variable entropy. The consistency relations are obtained in the general case. As an illustration, one-dimensional gas-dynamics are discussed and it is shown how the consistency relations are manipulated. The solution in which one or other of the Riemann variables is constant is obtained as a special case and motions of variable entropy are also attained.

1. Introduction. It is well-known that Einstein's theory of general relativity is a generalization of the Newtonian mechanics of a continuous fluid but, as far as the present author is aware, it has not hitherto been realised that Einstein's theory can serve as a tool in classical gas-dynamics. The object of the present paper is to show how this comes about. The solutions of the equations governing the motion of a gas which are obtained impose no limitations on the magnitude of the gas-velocity nor do they pre-suppose that adiabatic conditions prevail. But they do imply that the gas is perfect and non-viscous and that its motion takes place under the influence of its pressure-gradient alone.

The Newtonian absolute time will be denoted by T and the rectangular coordinates in Newtonian absolute space by (X_1, X_2, X_3). The three velocity components of the gas will be written (U_1, U_2, U_3). The summation convention will be used throughout, repeated greek indices running through the values 1 to 4, whilst repeated latin indices will take the values 1 to 3. The letters l, m, n will stand for any cyclic permutation of the numbers 1, 2, 3. With these conventions, the classical equations of motion of a gas are

$$\frac{\partial U_i}{\partial T} + U_j \frac{\partial U_i}{\partial X_j} = -\frac{1}{\rho}\frac{\partial p}{\partial X_i} + F_i, \qquad (i = 1, 2, 3),$$

where (F_1, F_2, F_3) is the force per unit mass of gas, excluding the pressure-gradient force; and the equation of continuity is

$$\frac{\partial \rho}{\partial T} + \frac{\partial \rho U_i}{\partial X_i} = 0. \tag{1.01}$$

The equations of motion can, with the aid of the continuity equation, also be written as

$$\frac{\partial}{\partial T}(\rho U_i) + \frac{\partial}{\partial X_j}(\rho U_i U_j + \delta_{ij} p) = \rho F_i, \qquad (i = 1, 2, 3) \tag{1.02}$$

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where δ_{ij} is the Kroenecker delta. The four equations (1.01) and (1.02) express the conservation of momentum and of mass in classical gas-dynamics. The thermodynamical quantities usually associated with a gas-motion are the velocity of sound, a, and the entropy of unit mass of gas, which may be defined as follows: If the specific heats at constant pressure and at constant volume are c_p and c_v , respectively, and their ratio, c_p/c_v , is denoted by k, then

$$a^2 = k\frac{p}{\rho}, \tag{1.03}$$

The entropy, S, of unit mass of a perfect gas, whose temperature is θ and whose equation of state is $p = R\rho\theta$, is most conveniently defined thus: Let dQ/dT be the rate of change "following the motion" of the heat-content of unit mass of gas, then

$$J\frac{dQ}{dT} = Jc_v\frac{d\theta}{dT} + p\frac{d}{dT}\left(\frac{1}{\rho}\right), \tag{1.04}$$

whence, using the equation of state and the relations

$$R = J(c_p - c_v) = (k - 1)Jc_v,$$

it follows that

$$\frac{1}{\theta}\frac{dQ}{dT} = c_v\frac{d}{dT}(\log p - k \log \rho).$$

In thermodynamics, the differential of the entropy is defined by dQ/θ, but, for our purposes, it is more convenient to write

$$dS = dQ/(c_v\theta).$$

Hence

$$\frac{dS}{dT} = \frac{d}{dT}\{\log p - k \log \rho\}, \tag{1.05}$$

or

$$S = \log \kappa + \log p - k \log \rho, \tag{1.06}$$

where κ is a constant, characteristic of each separate unit mass of gas. In particular, if all unit masses have the same constant value of κ, the pressure and density of the gas are related by the isentropic relation

$$p = \kappa\rho^k. \tag{1.07}$$

It is now necessary to summarize briefly the notions of general relativity which will be required*, but the reader who is not interested in this aspect of the matter may pass directly to Equations (2.06) to (2.09) and content himself with the process of verification described in the text immediately following these equations. In general relativity, an event is specified by four coordinates (x^4, x^1, x^2, x^3) of which the first denotes the time, and the other three, the place, of the occurrence in question. The

*For a more detailed treatment see, e.g. R. C. Tolman *Relativity, Thermodynamics and Cosmology*, Clarendon Press, Oxford, 1934.

events which, for example, constitute the history of the motion of a perfect gas are regarded as mapped in a four-dimensional Riemannian space-time whose metric is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu.$$

The velocity of the gas is represented by a four-dimensional vector (u^4, u^1, u^2, u^3) satisfying the equation

$$1 = g_{\mu\nu} u^\mu u^\nu, \tag{1.08}$$

whilst the mechanical quantities that are the counterparts of those whose partial derivatives appear in the equations (1.01) and (1.02), are the components of the energy-tensor

$$T^{\mu\nu} = \rho u^\mu u^\nu - g^{\mu\nu} \frac{p}{c^2}, \tag{1.09}$$

where ρ, p are the (invariant) density and pressure, $g^{\mu\nu}$ are the contravariant components of the metrical tensor $g_{\mu\nu}$, and c is the velocity of light. The energy-tensor and the metrical tensor are connected by the ten Einstein equations, viz.

$$-8\pi\gamma T^\mu_\nu = G^\mu_\nu - \tfrac{1}{2}\delta^\mu_\nu G \tag{1.10}$$

where γ is the constant of gravitation, $T^\mu_\nu = g_{\nu\lambda} T_{\mu\lambda}$, G^μ_ν is the contracted Riemann-Christoffel tensor and G, the invariant curvature. The last two tensors can be expressed in terms of the $g_{\mu\nu}$ and their first and second derivatives with respect to the coordinates, by the rules of the tensor calculus.

2. Newtonian approximation to Einstein's equations. It is sufficient for our purpose to consider orthogonal space-times whose coefficients differ only slightly from those of the space-time of special relativity where

$$g_{44} = 1, \qquad g_{11} = g_{22} = g_{33} = -\frac{1}{c^2}, \qquad g_{\mu\nu} = 0 \qquad (\mu \neq \nu).$$

We write $\epsilon = 2\gamma/c^2$ and assume throughout the calculations that terms of order ϵ higher than the first are negligible. An orthogonal space-time of the required kind has metric

$$ds^2 = D(dx^4)^2 - A(dx^1)^2 - B(dx^2)^2 - C(dx^3)^2 \tag{2.01}$$

where

$$\left.\begin{aligned} D &= 1 - 4\pi\epsilon\varphi, \\ A &= \frac{1}{c^2}\left\{1 + 4\pi\epsilon\left(\varphi + \frac{2\varphi_1}{c^2}\right)\right\} \\ B &= \frac{1}{c^2}\left\{1 + 4\pi\epsilon\left(\varphi + \frac{2\varphi_2}{c^2}\right)\right\}, \\ C &= \frac{1}{c^2}\left\{1 + 4\pi\epsilon\left(\varphi + \frac{2\varphi_3}{c^2}\right)\right\}, \end{aligned}\right\} \tag{2.02}$$

and φ, φ_1 , φ_2 , φ_3 are functions of all four coordinates (x^4 , x^1, x^2, x^3). The right-hand sides of (1.10) have been calculated by Dingle* for general values of D, A, B, C. If terms

*H. Dingle, Nat. Acad. Sci., Proc. 19, 559-563 (1933). Also given in R. C. Tolman, loc. cit., pp. 253-257.

of order ϵ are alone retained in his formulae, the following approximate forms of Einstein's equations are obtained:

$$\left.\begin{aligned} 4\pi\epsilon T^{lm} &= -4\pi\epsilon \frac{\partial^2\varphi_n}{\partial x^l\,\partial x^m}, \\ 4\pi\epsilon T^{l4} &= 4\pi\epsilon \frac{\partial^2}{\partial x^l\,\partial x^4}\left\{\varphi + \frac{1}{c^2}(\varphi_m + \varphi_n)\right\}, \\ 4\pi\epsilon T^{ll} &= 4\pi\epsilon\left\{-\frac{\partial^2}{(\partial x^4)^2}\left(\varphi + \frac{\varphi_m + \varphi_n}{c^2}\right) + \frac{\partial^2\varphi_m}{(\partial x^n)^2} + \frac{\partial^2\varphi_n}{(\partial x^m)^2}\right\}, \\ 4\pi\epsilon T^{44} &= -4\pi\epsilon\left\{\nabla^2\varphi + \frac{1}{c^2}\sum_{l,m,n}\left(\frac{\partial^2}{(\partial x^m)^2} + \frac{\partial^2}{(\partial x^n)^2}\right)\varphi_l\right\}, \end{aligned}\right\} \tag{2.03}$$

where

$$\nabla^2 = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2}.$$

Using also (1.09), the preceding equations become

$$\left.\begin{aligned} \rho u^l u^m &= -\frac{\partial^2\varphi_n}{\partial x^l\,\partial x^m}, \\ \rho u^l &= \frac{\partial^2}{\partial x^4\,\partial x^l}\left(\varphi + \frac{\varphi_m + \varphi_n}{c^2}\right), \\ \rho(u^l)^2 + p &= -\frac{\partial^2}{(\partial x^4)^2}\left(\varphi + \frac{\varphi_m + \varphi_n}{c^2}\right) + \frac{\partial^2\varphi_m}{(\partial x^n)^2} + \frac{\partial^2\varphi_n}{(\partial x^m)^2}, \\ \rho(u^4)^2 - \frac{p}{c^2} &= -\nabla^2\varphi - \frac{1}{c^2}\sum_{l,m,n}\left\{\frac{\partial^2}{(\partial x^m)^2} + \frac{\partial^2}{(\partial x^n)^2}\right\}\varphi_l, \end{aligned}\right\} \tag{2.04}$$

where, to a sufficient approximation, the velocity four-vector satisfies

$$(u^4)^2 - \frac{1}{c^2}\left\{(u^1)^2 + (u^2)^2 + (u^3)^2\right\} = 1. \tag{2.05}$$

There are eleven equations in the set (2.04) and (2.05), whose left-hand sides contain the six functions of the coordinates ρ, p, u^4, u^1, u^2, u^3; whilst their right-hand sides involve the four functions φ, φ_1, φ_2, φ_3. Obviously therefore there must be additional relationships between these two sets of functions and, in principle at least, it is possible to eliminate ρ, p, u^4, u^1, u^2, u^3 from (2.04) and (2.05) and thus obtain differential equations involving only φ, φ_1, φ_2, and φ_3. Such equations will be called *consistency relations* because, unless φ, φ_1, φ_2, φ_3 satisfy them, the eleven equations (2.04) and (2.05) will be mutually inconsistent. We shall show later how these consistency relations are obtained and how they are manipulated in the calculations: we must first derive from (2.04) and (2.05) the corresponding equations in Newtonian hydrodynamics. This is done, as usual in relativity theory, by neglecting terms of order $1/c^2$ and, to this end, it will be assumed that φ, φ_1, φ_2, φ_3, and the four u^α contain no terms of order c^2. The

Newtonian velocity components (U_1, U_2, U_3) will be the degenerate forms of the ratios $(dx^1/dx^4, dx^2/dx^4, dx^3/dx^4) = (u^1/u^4, u^2/u^4, u^3/u^4)$ when terms of order $1/c^2$ are neglected; and the coordinates (x^4, x^1, x^2, x^3) will become the Newtonian coordinates (T, X_1, X_2, X_3). Neglecting terms of order $1/c^2$ equation (2.05) reduces to the statement that $u^4 = 1$ whilst the following set of equations is obtained from (2.04):

$$\rho U_l U_m = -\frac{\partial^2 \varphi_n}{\partial X_l\, \partial X_m}, \tag{2.06}$$

$$\rho U_i = \frac{\partial^2 \varphi}{\partial X_i\, \partial T}, \qquad (i = 1, 2, 3), \tag{2.07}$$

$$\rho U_l^2 + p = -\frac{\partial^2 \varphi}{\partial T^2} + \frac{\partial^2 \varphi_n}{\partial X_m^2} + \frac{\partial^2 \varphi_m}{\partial X_n^2}, \tag{2.08}$$

$$\rho = -\nabla^2 \varphi. \tag{2.09}$$

In these ten equations ρ, p now stand for the Newtonian density and pressure respectively. It is easy to show by direct partial differentiation that, if the quantities given by (2.06) to (2.09) be substituted into (1.01) and (1.02), these equations are satisfied identically *provided that* $F_i = 0$, $(i = 1, 2, 3)$. Thus a solution of the equations of classical gas-dynamics has been obtained for the case when the gas is moving under the action of its pressure-gradient alone. The solution however involves the four functions $\varphi, \varphi_i (i = 1, 2, 3)$ of the coordinates X_i and of T, which are not independent but are subject to consistency relations that may be obtained as follows:

Elimination of ρ and the three U_i from equations (2.06), (2.07) and (2.09) yields the three equations

$$\frac{\partial^2 \varphi_l}{\partial X_m\, \partial X_n} = \left(\frac{\partial^2 \varphi}{\partial X_m\, \partial T} \cdot \frac{\partial^2 \varphi}{\partial X_n\, \partial T}\right) \Big/ \nabla^2 \varphi. \tag{2.10}$$

Again elimination of ρ, p and the three U_i from (2.07), (2.08) and (2.09) yields three equations, only two of which are however independent,

$$\left(\frac{\partial^2}{\partial X_l^2} - \frac{\partial^2}{\partial X_m^2}\right)\varphi_n + \frac{\partial^2}{\partial X_n^2}(\varphi_l - \varphi_m) = \left\{\left(\frac{\partial^2 \varphi}{\partial X_l\, \partial T}\right)^2 - \left(\frac{\partial^2 \varphi}{\partial X_m\, \partial T}\right)^2\right\} \Big/ \nabla^2 \varphi. \tag{2.11}$$

The ten equations (2.06) to (2.09) may then be replaced by the five consistency relations (2.10) and (2.11) together with the further five equations

$$\left.\begin{aligned} U_i &= -\frac{\partial^2 \varphi}{\partial X_i\, \partial T} \Big/ \nabla^2 \varphi, \qquad (i = 1, 2, 3), \\ \rho &= -\nabla^2 \varphi, \\ p &= -\frac{\partial^2 \varphi}{\partial T^2} + \frac{1}{3} \nabla^2\left(\sum_{i=1}^{3} \varphi_i\right) - \frac{1}{3} \sum_{i=1}^{3} \frac{\partial^2 \varphi_i}{\partial X_i^2} + \frac{1}{3} \sum_{i=1}^{3} \left(\frac{\partial^2 \varphi}{\partial X_i\, \partial T}\right)^2 \Big/ \nabla^2 \varphi, \end{aligned}\right\} \tag{2.12}$$

where, in the formula for p, it is presumed that φ_1, φ_2, φ_3 have been chosen so as to satisfy the consistency relations. It is worth noticing that ρ and $4\pi\varphi$ are connected by Poisson's equation and therefore $4\pi\varphi$ is the gravitational potential of the distribution of gas. The corresponding gravitational force has however been neglected in determining

the motion of the gas since, as we have seen, equations (2.06) to (2.09) imply that no force other than the pressure-gradient is acting.

3. One-dimensional gas-dynamics. As an illustration of the foregoing general theory, we consider certain types of motion in which the velocity of the fluid is parallel to a given straight line and in which the pressure and density vary spatially only with respect to distance measured parallel to this line. If the line is chosen to be the X_1-axis and distance along it be denoted by X, motions of this kind are defined by assuming that all the variables in equations (2.10), (2.11) and (2.12) are functions of X and T alone. It then follows at once from (2.12) that $U_2 = 0$, $U_3 = 0$, and that only $U_1 = U$ survives. The consistency relations (2.10) are identically satisfied, whilst the relations (2.11) are also satisfied by taking

$$\left.\begin{aligned} &\varphi_1 = 0, \qquad \varphi_2 = \varphi_3, \\ &\frac{\partial^2 \varphi_2}{\partial X^2} = \left(\frac{\partial^2 \varphi}{\partial X\, \partial T}\right)^2 \Big/ \frac{\partial^2 \varphi}{\partial X^2}. \end{aligned}\right\} \tag{3.01}$$

Hence formulae (2.12) reduce to

$$\left.\begin{aligned} U &= -\frac{\partial^2 \varphi}{\partial X\, \partial T} \Big/ \frac{\partial^2 \varphi}{\partial X^2}, \\ \rho &= -\frac{\partial^2 \varphi}{\partial X^2}, \\ p &= -\frac{\partial^2 \varphi}{\partial T^2} + \left(\frac{\partial^2 \varphi}{\partial X\, \partial T}\right)^2 \Big/ \frac{\partial^2 \varphi}{\partial X^2}, \end{aligned}\right\} \tag{3.02}$$

which provide the general solution of the problem of one-dimensional gas-dynamics which we are seeking by the method of Einstein's equations. The solution is given to within an arbitrary function φ of X and T, which, from the purely mathematical standpoint, can be chosen in any way we please. But it is important to notice that not every such selection will give a physically acceptable solution as may be seen by considering the choice

$$\varphi = A \cos (X - qT),$$

where A, q are constants, which leads through (3.02) to

$$U = q, \qquad \rho = A \cos (X - qT), \qquad p = 0.$$

Thus the velocity of the gas is constant, its pressure is zero and its density is alternately positive and negative, vanishing at places where $X = qT \pm (n + \frac{1}{2})\pi$, ($n$, an integer). Such a distribution of gas would clearly not be considered physically admissible.

The method of solution of equations (3.02) therefore consists of finding, by trial and error, a function φ that will correspond to a physically significant situation. The process of selection may be guided by some a priori requirement respecting the mathematical forms of U, p or ρ. As an example, let it be required to find all permissible densities and pressures for a perfect gas moving in such a way that its velocity is given by

$$U = (n + 1)[X/T + nq/(n + 1)], \tag{3.03}$$

where n is a pure number and q is a constant with the dimensions of velocity. Writing $\mu = \partial\varphi/\partial X$ and substituting from (3.03) into the first of equations (3.02), it follows that μ satisfies the first-order partial differential equation

$$\frac{\partial\mu}{\partial T} + (n+1)\left(\frac{X}{T} + \frac{nq}{n+1}\right)\frac{\partial\mu}{\partial X} = 0,$$

whence

$$\mu = \frac{\partial\varphi}{\partial X} = f\left\{\left(\frac{X}{T} + q\right)T^{-n}\right\},$$

where f is an arbitrary function of the argument $(X/T + q)T^{-n}$. It then follows that

$$\varphi = T^{n+1}F\left\{\left(\frac{X}{T} + q\right)T^{-n}\right\} + H(T), \tag{3.04}$$

where H is an arbitrary function of T and

$$F = \int f\left\{\left(\frac{X}{T} + q\right)T^{-n}\right\}T^{-(n+1)}\,dX,$$

T being treated as a constant in this integration. Introducing a new variable ζ by

$$\zeta = \left(\frac{X}{T} + q\right)T^{-n}, \tag{3.05}$$

and denoting differentiation with respect to T by a prime, equations (3.02) yield

$$\left.\begin{aligned} U &= (n+1)\{\zeta T^{n} - q/(n+1)\}, \\ \rho &= -T^{-(n+1)}\frac{d^2F}{\partial\zeta^2}, \\ p &= n(n+1)T^{n-1}\left\{\zeta^2\frac{d}{d\zeta}\left(\frac{F}{\zeta}\right) - \frac{H''T^{1-n}}{n(n+1)}\right\}. \end{aligned}\right\} \tag{3.06}$$

By means of these expressions for p and ρ, it is possible to calculate the velocity of sound in the gas and the rate of change of entropy of unit mass following its motion. Formula (1.03) gives

$$a^2 = k(n+1)n\left\{T^{2n}\zeta^2\frac{d}{d\zeta}\left(\frac{F}{\zeta}\right) - \frac{H''T^{1+n}}{n(n+1)}\right\}\bigg/\left(-\frac{d^2F}{d\zeta^2}\right), \tag{3.07}$$

whilst (1.05) with $d/dT = \partial/\partial T + U(\partial/\partial X)$ yields

$$\begin{aligned} \frac{dS}{dT} &= \frac{k(n+1)}{T} \\ &\quad + \frac{n-1}{T}\left\{\zeta^2\frac{d}{d\zeta}\left(\frac{F}{\zeta}\right) - \frac{H'''T^{2-n}}{(n+1)n(n-1)}\right\}\bigg/\left\{\zeta^2\frac{d}{d\zeta}\left(\frac{F}{\zeta}\right) - \frac{H''T^{1-n}}{(n+1)n}\right\}. \end{aligned} \tag{3.08}$$

Obviously, a physically acceptable case must have $\rho > 0$, i.e. F must be such that $d^2F/d\zeta^2 < 0$ for all values of ζ corresponding to the region of space occupied by the moving gas, and for all relevant times. Moreover the pressure p and the square of the velocity of sound must both be positive; hence F, H and n must be chosen so as to fulfill

these requirements. As an aid to the selection of F, H and n it is instructive to specialize formula (3.08) in two independent ways. Firstly, let it be assumed that H satisfies

$$H'' = (\text{constant})T^{n-1}, \tag{3.09}$$

where the constant may have the value zero. Equation (3.08) then reduces to

$$\frac{dS}{dT} = \frac{k(n+1) + (n-1)}{T}. \tag{3.10}$$

Alternatively, let it be assumed that each unit mass of gas conserves its entropy as it moves, a condition expressed by $dS/dT = 0$ or

$$\{k(n+1) + (n-1)\}\zeta^2 \frac{d}{d\zeta}\left(\frac{F}{\zeta}\right) - \frac{T^{1-n}}{(n+1)n}\{k(n+1)H'' + TH'''\} = 0. \tag{3.11}$$

If both (3.09) and (3.11) hold simultaneously, and F is arbitrary, n is determined in terms of k by

$$n = -(k-1)/(k+1), \tag{3.12}$$

and since for a real gas

$$2 > k > 1, \tag{3.13}$$

n must be a negative number.

(i) Velocity of Sound a Linear Function of X/T.

The effect of choosing particular mathematical expressions for F must now be investigated and, as a first example, suppose that

$$F = -A\zeta^{\lambda} \tag{3.14}$$

where $A(> 0)$ and λ are constants; and suppose also that (3.09) is satisfied by taking $H \equiv 0$. Then (3.06) and (3.07) become

$$\left.\begin{aligned}
U &= (n+1)\{\zeta T^n - q/(n+1)\} = (n+1)\{X/T + nq/(n+1)\},\\
\rho &= A\lambda(\lambda-1)\zeta^{\lambda-2}T^{-(n+1)} = A\lambda(\lambda-1)(X/T+q)^{\lambda-2}T^{-n(\lambda-1)-1},\\
p &= -A(\lambda-1)n(n+1)\zeta^{\lambda}T^{n-1} = -A(\lambda-1)n(n+1)(X/T+q)^{\lambda}T^{-n(\lambda-1)-1},\\
a &= \left\{-\frac{k}{\lambda}n(n+1)\zeta^2T^{2n}\right\}^{1/2} = \left\{-\frac{k}{\lambda}n(n+1)\right\}^{1/2}(X/T+q),
\end{aligned}\right\} \tag{3.15}$$

whilst dS/dT is given by (3.10). Thus the choice (3.14) for F leads to an expression for the velocity of sound which, like that for U, is linear in X/T. But p, ρ and a^2 must all be positive in a physically acceptable solution and therefore

$$n(n+1) < 0 \quad \text{and} \quad \lambda > 1. \tag{3.16}$$

The first of these conditions is clearly satisfied in the constant entropy case in which n is given by (3.12). If further, the motion is *isentropic* and the expressions (3.15) for p and ρ are substituted into (1.07), it follows, by considering the indices of ζ and T

in the resulting formula, that

$$k(\lambda - 2) = \lambda \qquad \text{and} \qquad k(n + 1) = 1 - n.$$

The second condition is, of course, the same as (3.12) whilst the first determines λ in terms of k, viz.

$$\lambda = 2k/(k - 1). \tag{3.17}$$

Because of (3.13), λ is greater than unity as required by (3.16). Using (3.12), (3.17) and (3.15), the solution for isentropic motion is

$$\left.\begin{aligned} U &= \frac{2}{k+1}\left(\frac{X}{T} - \frac{k-1}{2} q\right), \\ \rho &= \frac{2k(k+1)A}{(k-1)^2}\left(\frac{X}{T} + q\right)^{2/(k-1)}, \\ p &= \frac{2A}{k+1}\left(\frac{X}{T} + q\right)^{2k/(k-1)}, \\ a &= \frac{k-1}{k+1}\left(\frac{X}{T} + q\right). \end{aligned}\right\} \tag{3.18}$$

In the conventional treatment of isentropic motion in gas-dynamics based on the method of Riemann*, the solution (3.18) is obtained by putting one of the Riemann variables r, s equal to a constant. But the Riemann method gives, in the first instance, explicit values of U and of a, whereas the present one simultaneously determines p, ρ, U and a. This is due to the use of the gravitational potential $4\pi\varphi$ that corresponds to the density ρ. A second point of difference is that the isentropic condition (1.07) is introduced a priori in Riemann's method and it is therefore not easy to modify the method when variable entropy motions are in question. Such cases arise, for example, in the motions of interstellar gas clouds which are losing energy by radiation, a problem that has been attempted by Burgers† using the classical treatment. Variable entropy motions however present no greater difficulty than do adiabatic motions if the method we are here presenting be employed. For example, the solution (3.15) with the conditions (3.16) corresponds in general to variable entropy, the rate of change of entropy following the motion falling off inversely with the time by (3.10). In such motions the rate of loss of internal heat-energy in ergs per sec. per cm^3 of the gas is

$$\frac{dE}{dT} = -J\rho\frac{dQ}{dT} = -\frac{R\rho\theta}{k-1}\frac{dS}{dT} = -\frac{p}{k-1}\frac{dS}{dT}.$$

Using (3.10), and (3.15) there comes

$$\frac{dE}{dT} = \frac{-A(\lambda - 1)}{k-1}\{-n(n+1)\}\{k(n+1) + (n-1)\}\left(\frac{X}{T} + q\right)^{\lambda} T^{-n(\lambda-1)-2}. \tag{3.19}$$

*B. Riemann, *Oeuvres Mathématiques*, Paris, p. 177, 1898. A summary of the method is given in G. C. McVittie, Mon. Not. Roy. Astron. Soc., London **110**, 224-237, (1950).

†J. M. Burgers, K. Ned. Akad. v. Wet., **29**, 600 (1946).

By hypothesis $A > 0$, and, by (3.13) and (3.16), $2 > k > 1$, $\lambda > 1$, and $-1 < n < 0$; hence all the constant factors in dE/dT are positive provided that $k(n + 1) + (n - 1)$ is also positive. If the gas is monatomic, $k = 5/3$, and n must lie in the restricted range $-1/4 < n < 0$; if the gas is diatomic, $k = 7/5$, and n lies in $-1/6 < n < 0$; and so on. Thus by a suitable choice of n depending on the value of k, equations (3.15) give the motion of a gas which is losing heat-energy per unit volume at the rate given by (3.19).

(*ii*) *Velocity of Sound a Quadratic Function of* X/T.

As a second illustrative example in the choice of F and H, we consider

$$F = -A\zeta \log (\zeta + B), \qquad H'' = 4ABn(n + 1)T^{n-1}, \qquad \frac{dS}{dT} = 0,$$

where $A(> 0)$ and B are constants. Equations (3.06) and (3.07) then become

$$U = (n + 1)\{X/T + ng/(n + 1)\},$$

$$\rho = \frac{A}{T} (X/T + q + 2BT^{n})/(X/T + q + BT^{n})^2,$$

$$p = -n(n + 1) \frac{A}{T} (X/T + q + 2BT^{n})^2/(X/T + q + BT^{n}),$$

$$a^2 = -n(n + 1)k(X/T + q + 2BT^{n})(X/T + q + BT^{n}),$$

where $n = -(k - 1)/(k + 1)$. Since $n < 0$ for any real gas, it is evident that, as T increases, U and a will ultimately both be linear functions of $(X/T + q)$, as in (3.18), but the formulae for ρ and p will not become functions of $(X/T + q)$ alone.

4. Further developments and conclusions. The one-dimensional gas-dynamics discussed in the preceding section do not exhaust the applications of the equations (2.10), (2.11) and (2.12). The case of spherical masses of gas in motion has also proved tractable and, by proceeding to the second approximation to Einstein's equations through the inclusion of terms in ϵ^2, it has been possible to take account of the gravitational self-attraction of the mass of gas. The initial coordinate system employed in (2.01) is moreover not unique and an analysis of the different permissible coordinate systems has thrown light on the meaning of coordinate systems in general relativity and on the problem of gravitational waves in that theory. These and other related problems will be discussed by the author in a forthcoming publication. In the meantime, it has seemed appropriate to give the essence of the method in the hope that it will be found useful by other investigators.

—NOTES—

LARGE DEFLECTIONS OF A CANTILEVER BEAM WITH UNIFORMLY DISTRIBUTED LOAD*

By F. VIRGINIA ROHDE (*University of Florida*)

In this note only beams of uniform cross section are considered. The basic assumptions are that the deformations are elastic and that the bending does not alter the length of the beam.

Barten [1, 2] and Bisshopp and Drucker [3] have obtained results for a concentrated load. In the case of the uniformly distributed load, the Bernoulli-Euler equation,

$$\frac{1}{\rho} = \frac{d\theta}{ds} = \frac{M}{EI},$$

gives, with the origin at the free end of the beam,

$$\frac{dM}{ds} = -ws \cos \theta = EI \frac{d^2\theta}{ds^2}.$$

This equation does not lend itself to any simple solution. Put

$$\theta = \sum_{n=0}^{\infty} a_n s^n; \qquad \frac{d\theta}{ds} = \sum_{n=1}^{\infty} n a_n s^{n-1}; \qquad \frac{d^2\theta}{ds^2} = \sum_{n=2}^{\infty} n(n-1) a_n s^{n-2}.$$

The boundary conditions $s = 0$, $\theta = \alpha$, $d\theta/ds = 0$ give $a_0 = \alpha$; $a_1 = 0$. Then

$$\frac{d^2\theta}{ds^2} = -\frac{ws}{EI}(\cos\alpha \cos T - \sin\alpha \sin T), \qquad T = \sum_{n=2}^{\infty} a_n s^n.$$

Expanding $\cos T$ and $\sin T$ and comparing coefficients gives

$$a_{3n+4} = a_{3n+5} = 0, \qquad n = 0, 1, 2, \cdots; \qquad a_2 = 0;$$

$$a_3 = -w \cos \alpha / 6EI; \qquad a_6 = a_3 w \sin \alpha / 30EI;$$

etc. Thus

$$\theta = \alpha + \sum_{k=1}^{\infty} a_{3k} s^{3k}.$$

Since $dy/ds = \sin \theta$,

$$y = \int_0^s \sin \theta \, ds = T_1 \sin \alpha + T_2 \cos \alpha; \tag{1}$$

$$T_1 = s - a_3^2 s^7/14 - a_3 a_6 s^{10}/10 - \cdots;$$

$$T_2 = a_3 s^4/4 + a_6 s^7/7 + (a_9 - a_3^3/6) s^{10}/10 + \cdots.$$

Similarly,

$$x = T_1 \cos \alpha - T_2 \sin \alpha. \tag{2}$$

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Finally,

$$M = EI \sum_{k=1}^{\infty} 3ka_{3k}s^{3k-1};$$

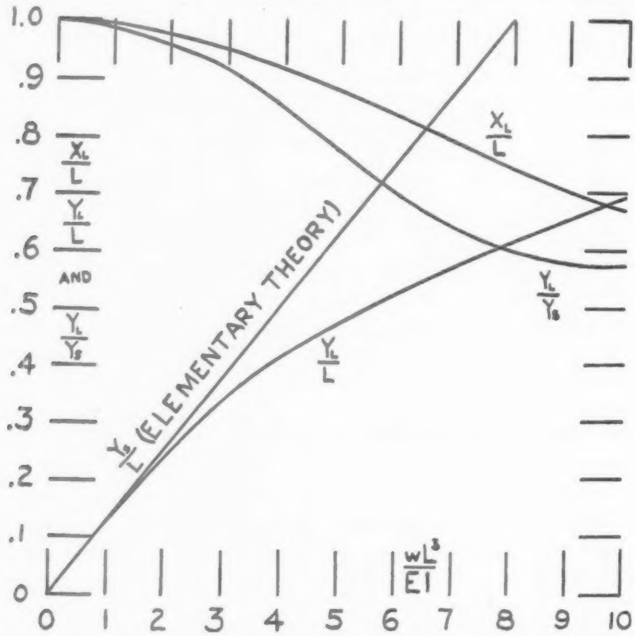
$$\alpha = -\sum_{k=1}^{\infty} a_{3k}L^{3k}. \tag{3}$$

With $s = L$, $\alpha = wL^3/6EI$, the first approximation to the maximum deflection is

$$y \doteq s \sin \alpha + \frac{1}{4} a_3 s^4 \cos \alpha \doteq wL^4/8EI,$$

which is in agreement with small deflection theory.

For computation, various values of α are chosen arbitrarily and corresponding values of wL^3/EI are found by eqn. (3). Against these values are plotted the ratios x_L/L, the horizontal projection of the deflection curve to the total length; y_L/L, the maximum deflection to the total length; and y_L/y_e, the maximum deflection to the maximum deflection as found by elementary theory. The results are shown in the figure.



For $wL^3/EI < 2$, values obtained by small deflection theory agree quite well with those obtained by large deflection theory. It is to be noted that the use of small deflection theory amounts to increasing the factor of safety.

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NOTE ON RECTANGULAR PLATES: DEFLECTION UNDER PYRAMIDAL LOAD*

By WILHELM ORNSTEIN (*Newark College of Engineering*)

Using the Euler-Fourier method, a direct procedure leading to the evaluation of Fourier coefficients in the computation of the deflection of thin plates is developed.

Consider a thin rectangular plate of uniform thickness, placed horizontally on four supports and acted upon by an arbitrary distributed load (Fig. 1). To solve the well known differential equation of the rectangular plate,

$$\frac{EI}{1-\mu^2}\nabla^2\nabla^2 w = p(x, y), \tag{1}$$

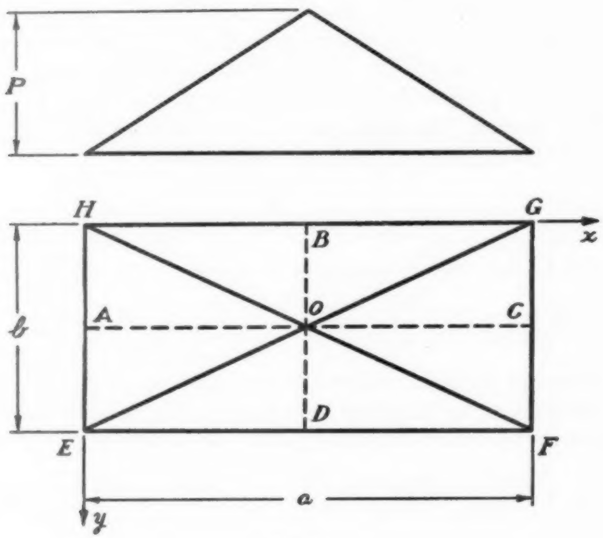

FIG. 1.

the deflection is represented in a form of a double infinite series whose every term satisfies the boundary conditions:

$$w = \sum_m \sum_n A_{mn} \sin m\pi\frac{x}{a} \sin n\pi\frac{y}{b}. \tag{2}$$

Substituting this expression in the plate equation (1), we obtain

$$\sum_m \sum_n A_{mn}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \sin m\pi\frac{x}{a} \sin n\pi\frac{y}{b} = \frac{1-\mu^2}{\pi^4 EI} p(x, y). \tag{3}$$

Multiplying both sides of equation (3) by $\sin (m'\pi x/a)\cdot dx$ and integrating from 0 to a and then multiplying both sides of the equation by $\sin (n'\pi y/b)\cdot dy$ and integrating

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from 0 to b we obtain

$$A_{mn} = \frac{4(1 - \mu^2)}{\pi^4 EIab}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2} \int_0^b \left[\int_0^a p(x, y) \sin m\pi \frac{x}{a}\, dx\right] \sin n\pi \frac{y}{b}\, dy. \qquad (4)$$

The expression (4) has been developed for any arbitrary, continuously distributed load, but it may, with small modifications, be used for a concentrated load.

Application to pyramidal distribution of load. The load is symmetrical with respect to two central axes AC and BD, see Fig. 1. If loads of equal magnitude are applied at two points equidistant from the axis BD, at $P_1(x_1, y_1)$ and at $P_2(x_2, y_1)$ then it is obvious that due to symmetry $x_1 + x_2 = a$ and

$$\sin (m\pi x_1/a) = \sin m\pi(1 - x_2/a) = \sin (m\pi x_2/a) \qquad (5)$$

and similarly $\sin (n\pi y_1/b) = \sin (n\pi y_2/b)$. Starting with the determination of A_{mn} due to the loads OEH and OFG, we see that when the contribution of the partial load OAH is known, then it is only necessary to multiply that value by four to have the expression for A_{mn} due to loads OEH and OFG. By the same reasoning, when we multiply by four the contribution made by load OHB to A_{mn}, the value of A_{mn} due to loads OHG and OEF is obtained. The maximum load on the plate is at the center, where its value is P. At any point (x,y) on OAH the load is

$$p = \frac{2Px}{a}, \qquad (6)$$

i.e., a function of x alone. Substituting this value of p in the expression (4), we obtain

$$A_{mn} = \frac{8P(1 - \mu^2)}{\pi^4 EIab}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2} \int_0^{a/2} \left[\int_{y_0}^{b/2} \sin n\pi \frac{y}{b}\, dy\right] \frac{x}{a} \sin m\pi \frac{x}{a}\, dx. \qquad (7)$$

In the second integral of the expression (7) the lower limit y_0 is equal to $y_0 = bx/a$ which is the equation of the diagonal HF. Because of symmetry of the load, both m and n are odd numbers, so that after integration the expression (7) takes the form

$$A_{mn} = \frac{8P(1 - \mu^2)}{\pi^5 EIn}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2} \int_0^{1/2} \frac{x}{a} \sin m\pi \frac{x}{a} \cos n\pi \frac{x}{a}\, d\left(\frac{x}{a}\right). \qquad (8)$$

Setting $\pi x/a = u$, the integral of this expression becomes

$$\frac{1}{\pi^2}\int_0^{\pi/2} u \sin (mu) \cos (nu)\, du$$

$$= \frac{1}{2\pi^2}\int_0^{\pi/2} u \sin (m + n)u\, du + \frac{1}{2\pi^2}\int_0^{\pi/2} u \sin (m - n)u\, du. \qquad (9)$$

Integrating by parts, we obtain the value of A_{mn} for the load OAH

$$A_{mn} = \frac{-2P(1 - \mu^2)}{\pi^6 EI}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2}\left[\frac{\cos [(m + n)\pi/2]}{n(m + n)} + \frac{\cos [(m - n)\pi/2]}{n(m - n)}\right]. \qquad (10)$$

Interchanging a with b and m with n, the value of A_{mn} for the load OHB is obtained:

$$A_{mn} = \frac{-2P(1 - \mu^2)}{\pi^6 EI}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2}\left[\frac{\cos [(m + n)\pi/2]}{m(m + n)} + \frac{\cos [(n - m)\pi/2]}{m(n - m)}\right]. \qquad (11)$$

Adding to the expression (10) the contributions made by the loads OAE, OCG and OCF, and further, adding to the expression (11) the influence of loads OBG, ODE and ODF, the term A_{mn} for the whole plate is

$$A_{mn} = \frac{-16P(1-\mu^2)}{\pi^6 EI mn}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2} \cos(m\pi/2)\cos(n\pi/2). \tag{12}$$

The expression (11) gives a trivial value of A_{mn}, because with both m and n odd, A_{mn} would vanish, which of course is impossible. It is seen that the equations (10) and (11) are true only for $m \neq n$, which, however, does not give any practical result. Moreover, we cannot set $m = n$, in the expression (10) and (11) because a value A_{mn} equal to infinity would result. It is clear that the integration performed is true only when m equals n; hence we must go back to the expression (9) and set there $m = n$. With this substitution expression (9) will yield the following value

$$\frac{1}{2\pi^2}\int_0^{\pi/2} u \sin 2mu\, du = \frac{1}{4m\pi}. \tag{13}$$

For the load OAH, the term A_{mn} from (8) becomes

$$A_{mn} = \frac{2P(1-\mu^2)}{\pi^6 m^6 EI}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{-2} \tag{14}$$

and for the whole plate

$$A_{mn} = \frac{16P(1-\mu^2)}{\pi^6 m^6 EI}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{-2}. \tag{15}$$

Finally the expression for the deflection is

$$w = \frac{16P(1-\mu^2)}{\pi^6 EI}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{-2}\left\{\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}+\frac{1}{3^6}\sin\frac{3\pi x}{a}\sin\frac{3\pi y}{b}+\cdots\right\}. \tag{16}$$

Thus, through this operation a double series for the deflection of the rectangular plate under pyramidal load is reduced to a result involving but a single series.

A RANDOM WALK RELATED TO THE CAPACITANCE OF THE CIRCULAR PLATE CONDENSER*

By E. REICH (*The RAND Corporation, Santa Monica, Cal.*)

Abstract. It is shown that the solution of Love's equation for the capacitance of the circular plate condenser can be expressed in terms of the mean duration of a certain one-dimensional random walk with absorbing barriers. The interpretation as a random walk makes it possible to confirm the fact that the actual capacitance of the condenser is always larger than the value given by the standard approximation for small separations, and yields an upper bound as well. In addition to its theoretical interest, the

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random walk appears to provide a practical means for the calculation of the capacitance by a Monte Carlo technique.

1. Introduction. The purpose of this note is to point out an equivalence between the mean number of steps of a random walk, and the capacitance C of a condenser consisting of two equal, infinitely thin circular conducting disks of radius a, separated by a distance κa.

It is a classical result that for $\kappa \to 0$, $C \sim a/4\kappa$, and it is also known (Jeans [1]) that $\lim_{\kappa\to\infty} C = a/\pi$. A more precise formula for small κ dates back to Kirchhoff [3], who also discusses a paper by Clausius on this subject. A formula of Nicholson's [5] said to give C for all κ by means of a definite integral has turned out to be incorrect. (Love [4] has pointed out fallacies in Nicholson's reasoning. Moreover, it can be shown by direct consideration of Nicholson's integral that it does not even give the correct asymptotic formula as $\kappa \to 0$). Present-day knowledge regarding the value of C for general values of κ appears to be restricted to the following elegant result obtained by Love.

LEMMA 1. *If $f(t)$ is the solution of the integral equation*

$$f(x) - \frac{1}{\pi}\int_{-1/\kappa}^{1/\kappa} \frac{f(t)}{1 + (x - t)^2}\, dt = 1 \qquad (-1/\kappa \le x \le 1/\kappa) \tag{1}$$

then

$$C = \frac{a\mu}{\pi}, \tag{2}$$

where

$$\mu = \frac{\kappa}{2}\int_{-1/\kappa}^{1/\kappa} f(t)\, dt. \tag{3}$$

It will be shown that both $f(t)$ and μ have very simple interpretations in terms of a random walk in the interval $|x| < 1/\kappa$. The results are expressed by the following theorem.

THEOREM 1. Let $\theta_i (i = 1, 2, \cdots)$ be independent random variables, such that $0 \le \theta_i \le 2\pi$, and Prob $\{\theta_i < A\} = 1/2\pi\, A$, $(0 \le A \le 2\pi)$. Consider the random walk starting at $x \in [-1/\kappa, 1/\kappa]$ whose i-th step equals $\tan \theta_i$. We will say that absorption occurs at step k if the k-th step is the first step landing outside $[-1/\kappa, 1/\kappa]$. *Then $f(x)$ is the expected number of steps to absorption.*

COROLLARY. Let the walk start at a random point in $[-1/\kappa, 1/\kappa]$ (i.e., a point drawn from a distribution with flat density). *Then μ is the expected number of steps to absorption.*

2. Derivation. It will be seen that Theorem 1 is a corollary of the following result:

THEOREM 2*. If

(a) $K(x,t)$ is continuous and ≥ 0 $(a \le x \le b, a \le t \le b)$,

(b) there exists a constant $\alpha > 0$ such that

$$\int_a^b K(x,t)\, dt \le 1 - \alpha \qquad \text{for all } x \in [a,b], \tag{4}$$

*See Wasow [6] for a different proof of this result under a less restrictive hypothesis.

then there exists a random walk starting at x whose mean number of steps to absorption is $f(x)$, where

$$f(x) - \int_a^b K(x,t)f(t)\,dt = 1 \qquad (a \leq x \leq b). \tag{5}$$

The dispersion of the number of steps will be finite.

First we derive the rather obvious

LEMMA 2. If ξ and η are random variables, taking on non-negative integral values, and

$$P_k = \text{Prob}\,\{\xi \geq k\} \leq Q_k = \text{Prob}\,\{\eta \geq k\} \qquad (k = 0, 1, 2, \cdots)$$

then

$$\langle \xi \rangle \leq \langle \eta \rangle \qquad \text{and} \qquad \langle \xi^2 \rangle \leq \langle \eta^2 \rangle.$$

Proof:

$$\langle \eta \rangle = \sum_0^\infty k(Q_k - Q_{k+1}) \geq \sum_0^N k(Q_k - Q_{k+1}) + (N+1)Q_{N+1} = \sum_1^{N+1} Q_k\,.$$

Letting $N \to \infty$, we see that the term to the right of the inequality approaches $\langle \eta \rangle$, showing that $\langle \eta \rangle = \sum_1^\infty Q_k$, and consequently $\langle \xi \rangle \leq \langle \eta \rangle$. The inequality for the second moments can be proved in the same way.

Proof of Theorem 2. For each $x \,\varepsilon\, [a,b]$ construct a continuous function $S(x,t)$ such that

(a) $S(x,t) \geq 0 \qquad (-\infty < t < \infty)$

(b) $\int_{-\infty}^{\infty} S(x,t)\,dt = 1$

(c) $S(x,t) = K(x,t) \qquad$ for $\qquad a \leq t \leq b.$

For fixed x, the function $S(x,t)$ should be thought of as a probability density. The random walk is defined inductively as follows: We start at x, and stop when first landing outside $[a,b]$. Having taken n steps, and assuming the nth step lands us at $y \,\varepsilon\, [a, b]$, the next step is taken to the point λ, where

$$\text{Prob}\,\{\lambda < u\} = \int_{-\infty}^{u} S(y,t)\,dt.$$

Let $\xi = \xi(x) =$ number of steps to absorption if the random walk starts at x, and $P_k = P_k(x) = \text{Prob}\,\{\xi \geq k\}$. If before a given step occurs, absorption has not yet taken place, then the probability of absorption as a result of the step is not smaller than α. Thus $P_1 = 1$, and $P_k \leq Q_k$, where $Q_k = (1-\alpha)^{k-1}$. By Lemma 2, $\langle \xi \rangle = f(x) \leq 1/\alpha$, and $\langle \xi^2 \rangle \leq (2-\alpha)/\alpha^2$. Dispersion $[\xi] = \langle (\xi - \langle \xi \rangle)^2 \rangle = \langle \xi^2 \rangle - \langle \xi \rangle^2 \leq (2-\alpha)/\alpha^2$. Thus $f(x)$ exists, and has a finite dispersion.

Now to show that $f(x)$ satisfies (5).

Define $f(x) = 0$ for $x \not\varepsilon\, [a,b]$. Then for all random walks whose first step carries to

y, the mean number of steps to absorption is

$$f(x \mid y) = 1 + f(y)$$

Therefore

$$\int_{-\infty}^{\infty} f(x \mid y) S(x,y)\, dy = \int_{-\infty}^{\infty} S(x,y)\, dy + \int_{-\infty}^{\infty} f(y) S(x,y)\, dy,$$

which reduces to

$$f(x) = 1 + \int_a^b f(y) K(x,y)\, dy$$

as was to be proved.

Theorem 2 follows from the observation that the tangent of a random number uniformly distributed in $(0,2\pi)$ has the probability density $(1/\pi)/(1 + x^2)$. The α of Theorem 2 can be taken as

$$1 - 1/\pi \int_{-1/\kappa}^{1/\kappa} \frac{dx}{1 + x^2} = (2/\pi) \text{ Arctan } \kappa.$$

Thus

$$f(x) \leq \frac{\pi}{2 \text{ Arctan } \kappa}, \qquad \text{and} \qquad C \leq \frac{a}{2 \text{ Arctan } \kappa} \qquad \text{for any } \kappa.$$

3. Further remarks. It is interesting to note that checks on the asymptotic values of C as $\kappa \to \infty$, and $\kappa \to 0$ are easily possible.

For $\kappa \to \infty$, we have of course $f(x) \to 1$, as can be seen either from the random walk, or the integral equation (1). Therefore $\mu \to 1$, and $C \to a/\pi$.

For $\kappa \to 0$ the formula $C \sim a/4\kappa$ can be obtained by using a result of Kac and Pollard [2] which also yields the conclusion that $C > a/4\kappa$. Kac and Pollard consider a *continuous* random motion on the x-axis, where if the particle is at point x_0 at time t_0, its position at time $t + t_0$ is given by a Cauchy distribution with semi-interquartile range t. We now make the observation that if one observes such a continuous random motion at the discrete time points $t = 0, 1, 2, \cdots$, and records the observed positions at these instants, the result is the generation of a discrete walk with exactly the properties defined in the hypothesis of Theorem 1.

Let $f^*(x)$ be the mean time to absorption of the *continuous* random motion, if it starts at x. As a simple consequence of the above observation we have $f(x) > f^*(x)$. Therefore

$$\mu > \kappa/2 \int_{-1/\kappa}^{1/\kappa} f^*(x)\, dx.$$

Using the result of Kac and Pollard for $E\{T(a,b,t)\}$ ([2], page 383) we obtain

$$f^*(x) = \left(\frac{1}{\kappa^2} - x^2\right)^{1/2}.$$

Therefore

$$C = \frac{a\mu}{\pi} > \frac{a\kappa}{2\pi} \int_{-1/\kappa}^{1/\kappa} f^*(x)\, dx = \frac{a}{4\kappa}. \tag{6}$$

In other words, C is always larger than the value predicted by the usual asymptotic formula for small separations. For $\kappa \to 0$ it is heuristically evident that $f(x) \sim f^*(x)$, so that $C \sim a/(4\kappa)$, as expected.†

A simple, plausible, "physical" proof of the inequality $C > a/4\kappa$ can be given.* C equals the ratio of charge per plate to potential difference between plates. The capacitor can be considered as being cut out of a pair of infinite parallel plates. Before the cut-away portions are removed the ratio of charge to potential difference is $a/4\kappa$. When the cut-away portions are removed each particle of charge moves away from the axis, and therefore the field intensity at each point of the axis is decreased below its previous value. Since the potential difference is the integral of this field intensity along the axis, the argument shows that the potential difference is reduced, and the capacitance therefore increased from $a/4\kappa$.

4. Practicability as a Monte Carlo method. If $\sigma = \langle(\xi - \mu)^2\rangle$ is the standard deviation of ξ, and n walks are taken, yielding $\xi_1, \xi_2, \cdots \xi_n$ as the observed values of ξ, then $\sigma/n^{1/2}$ will be the standard deviation of the empirical mean, and $\epsilon = \sigma/\mu n^{1/2}$ will be approximately equal to the ratio of standard deviation to the mean.

We will restrict ourselves to small κ, as it is conjectured that the n required for achieving a given ϵ steadily decreases as κ increases. For small κ, $\sigma < 2^{1/2}/\alpha \sim \pi/(2^{1/2}\kappa)$, $\mu > \pi/4\kappa$. Therefore, asymptotically for small κ,

$$\epsilon \leq (8/n)^{1/2}. \tag{7}$$

A numerical experiment, using $\kappa = 0.1$, and $n = 1000$ was made, using automatic computing machinery. The results were an empirical mean value of 10.101, and an ϵ of approximately 0.034. For $n = 1000$ the right-hand side of (7) yields $\epsilon \leq 0.09$.

5. Acknowledgment. The author wishes to thank Mr. P. Swerling for his helpful suggestions during the writing of this paper.

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†*Note added in proof:* For a stronger result see, G. Polya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, Princeton, 1951.

*Communicated to the author by Mr. R. H. Dishington.

ON THE STABILITY OF LAMINAR BOUNDARY LAYER FLOW*

BY SIN-I CHENG (*Princeton University*)

The stability of two dimensional small disturbances in laminar boundary layer flow has been extensively studied on the assumption that boundary layer flows are essentially parallel flows. The direct effect of the local pressure gradient on the calculation of the stability limits for incompressible boundary layer flow has been shown to be negligible under the approximation $(\alpha R)^{-1} \ll 1$, if the local velocity profile is used in the stability calculation. However, the assumption that the vertical velocity in the boundary layer flow plays negligible role has not received careful attention.

There is a qualitative argument that under the Prandtl boundary layer approximation, the variation of the mean flow properties in the x-direction within a few wave lengths of the disturbance is of the order of R^{-1}, which is negligibly small compared to unity. Therefore, the contributions of such terms due to x-gradients of pressure and temperature in the stability calculation can be neglected as higher order small quantities. This kind of argument should be investigated more closely so far as the vertical velocity component is concerned even though the vertical velocity component is a small quantity of the order of R^{-1}. The vertical velocity component produces a momentum transfer and an energy transfer across the boundary layer where both the disturbance quantities and the mean flow properties vary rapidly. The net effect of the transport processes may thus be much larger than the magnitude of the small agent that produces the transport processes. While the vertical velocity component of the flow and the gradients of the flow properties in the x-direction are small quantities of the same order the net effect of the former in the stability calculation will be shown to be much more important than that of the latter.

From the two dimensional forms of the equations of mass continuity, momentum and energy with variable viscosity and thermal conductivity coefficient and constant specific heats, the linearized system of partial differential equations for the amplitude functions of the periodic disturbances of the type exp $[i\alpha(x - ct)]$ is obtained with two independent spatial variables x and y. The fast varying, frequency dependent part of the disturbances as functions of x is represented by the factor exp $(i\alpha x)$. The decay or growth of the amplitudes of the disturbances in the x-direction is slow and in the first approximation, the x-gradients of the amplitude functions can be considered as independent of x. The system of disturbance equations can hence be considered as a set of ordinary differential equations for f, ϕ, π, r and θ with y as the only independent variable. Thus, we have the following equations:

Continuity:

$$(\phi' + if) + \frac{v}{\alpha}\frac{r'}{\rho} = -i(w - c)\frac{r}{\rho} - \frac{\rho'}{\rho}\phi - \frac{r}{\rho}\frac{v'}{\alpha} - \frac{1}{\alpha\rho}\frac{\partial}{\partial x}(\rho f + wr) \tag{1}$$

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First Momentum:

$$\alpha\rho[i(w-c)f + w'\phi] + \rho[fw_x + wf_x + vf'] + [ww_x + vw']r = -\frac{1}{\gamma M^2}(i\alpha\pi + \pi_x)$$

$$+ \frac{1}{R}\left[\left(\frac{4}{3}\mu_1 + \frac{2}{3}\mu_2\right)(f_{xx} + 2i\alpha f_x - \alpha^2 f) + \left(\frac{\mu_1}{3} + \frac{2}{3}\mu_2\right)\alpha(\phi_x' + i\alpha\varphi') + \mu_1 f''\right]$$

$$+ \frac{1}{R}\frac{d\mu_1}{dT}\left[\left(\frac{4}{3} + \frac{2}{3}\tau\right)T_x(f_x + i\alpha f) + \frac{2}{3}(\tau - 1)T_x\alpha\phi' + \alpha T'(\phi_x + i\alpha\phi) + T'f'\right]$$

$$+ \frac{1}{R}\frac{d\mu_1}{dT}\left[\left\{\left(\frac{4}{3} + \frac{2}{3}\tau\right)w_x + \frac{2}{3}(\tau - 1)v'\right\}(\theta_x + i\alpha\theta) + (v_x + w')\theta'\right]$$

$$+ \frac{1}{R}\frac{d\mu_1}{dT}\left[\left(\frac{4}{3} + \frac{2}{3}\tau\right)w_{xx} + w'' + \left(\frac{1}{3} + \frac{2}{3}\tau\right)v_x'\right]\theta \qquad (2)$$

Second Momentum:

$$\alpha^2\rho[i(w-c)\phi] + \alpha\rho(v'\phi + v\varphi') + \rho v_x f + \alpha\rho w\varphi_x + (wv_x + vv')r$$

$$= \frac{\mu_1}{R}\{[f_x' + i\alpha f' + \alpha\varphi_{xx} + 2i\alpha^2\phi_x - \alpha^3\phi]$$

$$+ \left(\frac{4}{3} + \frac{2}{3}\frac{\mu_2}{\mu_1}\right)\alpha\phi'' + \frac{2}{3}\left(\frac{\mu_2}{\mu_1} - 1\right)(f_x' + i\alpha f')\Big\}$$

$$+ \frac{1}{R}\frac{d\mu_1}{dT}\left\{T_x(f' + \alpha\phi_x + i\alpha^2\phi) + \left(\frac{4}{3} + \frac{2}{3}\tau\right)T'\alpha\phi' + \frac{2}{3}(\tau - 1)T'(f_x + i\alpha f)\right\}$$

$$+ \frac{1}{R}\frac{d\mu_1}{dT}\left\{(v_x + w')(\theta_x + i\alpha\theta) + \left(\frac{4}{3} + \frac{2}{3}\tau\right)v'\theta' + \frac{2}{3}(\tau - 1)w_x\theta'\right\}$$

$$+ \frac{1}{R}\frac{d\mu_1}{dT}\left\{v_{xx} + \left(\frac{4}{3} + \frac{2}{3}\tau\right)v'' + \left(\frac{1}{3} + \frac{2}{3}\tau\right)w_x'\right\}\theta - \frac{\pi'}{\gamma M^2} \qquad (3)$$

Energy:

$$\alpha\rho[i(w-c)\theta + T'\phi] + \rho[w\theta_x + v\theta' + T_x f] + (wT_x + vT')r$$

$$= -(\gamma - 1)[p(f_x + i\alpha f + \alpha\phi') + (w_x + v')\pi]$$

$$+ \mu_1\frac{\gamma(\gamma - 1)M^2}{R}\left\{\left(\frac{4}{3} + \frac{2}{3}\frac{\mu_2}{\mu_1}\right)2[w_x(f_x + i\alpha f) + \alpha v'\phi']\right.$$

$$\left. + \frac{4}{3}\left(\frac{\mu_2}{\mu_1} - 1\right)[w_x\alpha\phi' + v'(f_x + i\alpha f)] + 2(w' + v_x)(f' + \alpha\phi_x + i\alpha^2\phi)\right\}$$

$$+ \frac{d\mu_1}{dT}\frac{\gamma(\gamma - 1)M^2}{R}\left[\left(\frac{4}{3} + \frac{2}{3}\tau\right)(w_x^2 + v'^2) + \frac{4}{3}(\tau - 1)w_x v' + (w' + v_x)^2\right]\theta$$

$$+ \frac{\gamma\mu_1}{\sigma R}[\theta'' + \theta_{xx} + 2i\alpha\theta_x - \alpha^2\theta]$$

$$+ \frac{d\mu_1}{dT}\frac{\gamma}{\sigma R}[T''\theta + T_{xx}\theta + 2T_x'(\theta_x + i\alpha\theta) + 2T'\theta'] \qquad (4)$$

State:

$$\frac{\pi}{p} = \frac{r}{\rho} + \frac{\theta}{T}, \tag{5}$$

where $\tau = [(d\mu_2/dT)/(d\mu_1/dT)]$, all other symbols are adopted from reference 1.

For the convenience in reducing the equations the quantities Z_i are defined in a manner slightly different from those in reference 1 and following the well known procedure, these variables Z_i are expanded into series of $\epsilon = (\alpha R)^{1/3}$ as follows:

$$\begin{aligned}
Z_1 &= \quad f = x_1^{(0)} + \epsilon x_1^{(1)} + \cdots \\
\epsilon Z_2 &= \quad \epsilon f' = x_2^{(0)} + \epsilon x_2^{(1)} + \cdots \\
\epsilon^{-1} Z_3 &= \quad \epsilon^{-1}\phi = x_3^{(0)} + \epsilon x_3^{(1)} + \cdots \\
Z_4 &= \quad \phi' = x_4^{(0)} + \epsilon x_4^{(1)} + \cdots \\
Z_5 &= \quad \theta = x_5^{(0)} + \epsilon x_5^{(1)} + \cdots \\
\epsilon Z_6 &= \quad \epsilon\theta' = x_6^{(0)} + \epsilon x_6^{(1)} + \cdots \\
\epsilon^{-1} Z_7 &= \epsilon^{-1}\frac{\pi}{M^2} = x_7^{(0)} + \epsilon x_7^{(1)} + \cdots \\
Z_8 &= \quad \frac{\pi'}{M^2} = x_8^{(0)} + \epsilon x_8^{(1)} + \cdots
\end{aligned} \tag{6}$$

The mean flow velocity components w and v, the mean density and temperature are also expanded into Taylor series about the critical point y_c where $w(y_c) = c$. The expansion of v is written as:

$$\begin{aligned}
v &= \frac{1}{\rho}\int_0^y \frac{\partial}{\partial x}(\rho w)\, dy = \frac{1}{R}\cdot\frac{1}{\rho}\int_0^y \frac{\partial}{\partial \xi}(\rho w)\, dy \\
&= \alpha\epsilon^3[v_0 + v_1\cdot\epsilon\eta + v_2\cdot(\epsilon\eta)^2/2 + \cdots]
\end{aligned}$$

where

$$v_0 = \frac{1}{\rho_c}\left[\frac{\partial(\rho w)_c}{\partial\xi}\, y_c - \frac{\partial(\rho w)_c'}{\partial\xi}\, y_c^2/2 + \cdots\right]$$

$$v_1 = \frac{1}{\rho_c}\left[\frac{\partial(\rho w)_c}{\partial\xi} - \frac{\rho_c'}{\rho_c}\, v_0\right], \qquad v_2 = \cdots$$

are of the order of unity for both compressible and incompressible flow. Substituting all these expansions into the disturbance equations and collecting terms of the same power of ϵ, one obtains for the first approximation of the order of unity or ϵ^0:

$$\left\{\begin{aligned}
&\frac{dx_3^{(0)}}{d\eta} + ix_1^{(0)} = 0 \\
&\frac{d^2x_1^{(0)}}{d\eta^2} - \frac{w_c'}{\nu_{1c}}(i\eta x_1^{(0)} + x_3^{(0)}) - \frac{i}{\gamma\mu_{1c}}\, x_7^{(0)} = 0 \\
&\frac{dx_7^{(0)}}{d\eta} = 0 \\
&\frac{d^2x_5^{(0)}}{d\eta^2} - \frac{\sigma}{\nu_{1c}}\left[iw_c'\eta x_5^{(0)} + \left(T_c' - \frac{\gamma-1}{\gamma}\frac{p_c'}{\rho_c}\right)x_3^{(0)}\right]
\end{aligned}\right. \tag{8}$$

These equations are identical with those given in reference 1, and solutions can be obtained therefrom.

For the second approximation of the order of ϵ one obtains:

$$
\left\{
\begin{aligned}
&\frac{dx_3^{(1)}}{d\eta} + ix_1^{(1)} = \frac{iw_c'}{T_c}\,\eta x_5^{(0)} - \frac{\rho_c'}{\rho_c}\,x_3^{(0)} \\
&\frac{d^2x_1^{(1)}}{d\eta^2} - \frac{w_c'}{\nu_{1c}}\,(i\eta x_1^{(1)} + x_3^{(1)}) - \frac{i}{\gamma\mu_{1c}}\,x_7^{(1)} \\
&\qquad = \left\{\frac{w_c''}{\nu_{1c}} + \frac{w_c'}{\nu_{1c}}\left[\frac{\rho_c'}{\rho_c} - \left(\frac{d(\ln \mu_1)}{dT}\right)_c T_c'\right]\right\}\eta(i\eta x_1^{(0)} + x_3^{(0)}) - i\,\frac{w_c''}{\nu_{1c}}\,\frac{\eta^2}{2}\,x_1^{(0)} \\
&\qquad\quad - \frac{i}{\gamma\mu_{1c}}\left(\frac{d(\ln \mu_1)}{dT}\right)_c T_c'\eta x_7^{(0)} - \left(\frac{d(\ln \mu_1)}{dT}\right)_c (w_c'x_6^{(0)} + T_c'x_2^{(0)}) \\
&\qquad\quad + \frac{v_0}{\nu_{1c}}\,x_2^{(0)} \\
&\frac{dx_7^{(1)}}{d\eta} = 0 \\
&\frac{d^2x_5^{(1)}}{d\eta^2} - \frac{\sigma}{\nu_{1c}}\left[iw_c'\eta x_5^{(1)} + \left(T_c' - \frac{\gamma - 1}{\gamma}\,\frac{p_c'}{\rho_c}\right)x_3^{(1)}\right]
\end{aligned}
\right.
$$

$$
\begin{aligned}
&= \frac{\sigma}{\nu_{1c}}\left\{\left[\frac{\rho_c'}{\rho_c} - \left(\frac{d(\ln \mu_1)}{dT}\right)_c T_c'\right](iw_c'\eta^2 x_5^{(0)} + T_c'\eta x_3^{(0)}) + iw_c''\,\frac{\eta^2}{2}\,x_5^{(0)}\right. \\
&\quad \left. - \frac{\gamma - 1}{\gamma}\left[\frac{p_c''}{\rho_c} - \left(\frac{d(\ln \mu_1)}{dT}\right)_c T_c'\,\frac{p_c'}{\rho_c}\right]\eta x_3^{(0)} + T_c''\eta x_3^{(0)}\right\} \\
&\quad - \frac{\gamma - 1}{\gamma}\,\frac{\sigma}{\mu_{1c}}\,iw_c'\eta M^2 x_7^{(0)} - 2\sigma(\gamma - 1)M^2 w_c'x_2^{(0)} - \left(\frac{d(\ln \mu_1)}{dT}\right)_c 2T_c'x_6^{(0)} \\
&\quad + \frac{\sigma}{\nu_{1c}}\,v_0 x_6^{(0)}
\end{aligned}
\tag{9}
$$

The homogeneous parts of equations (9) are the same as equations (8), and the inhomogeneous parts are known functions in terms of the solutions of equations (8) and mean flow properties. The second approximation can therefore be obtained by quadrature. It is observed that the vertical velocity component enters into both the momentum and the energy relations in the system of equations (9), while the x-gradients do not at this approximation. Terms involving x-gradients enter only at the next approximation of the order of ϵ^2. Therefore, the effect of the vertical velocity component is more critical than the effect of the gradients in the x-direction in the stability calculation.

Since $x_{2j}^{(0)} = (dx_{1j}^{(0)}/d\eta) = 0$ for $j = 3, 4, 5$ and 6, it is only the two viscous solutions $x_{11}^{(1)}$ and $x_{12}^{(1)}$ or $x_{31}^{(1)}$ and $x_{32}^{(1)}$ that are dependent on the vertical velocity component. $x_{6j}^{(1)}$ are all v-dependent but they enter into the characteristic value problem as higher

order small quantities even in the compressible case. The major effect of the vertical velocity component in the determination of the stability boundary is through $x_{31}^{(1)}$ and $x_{32}^{(1)}$.

Suppose we take all x_{3j} from the ϵ series and consistently take all six solutions to the order of ϵ i.e. $x_{3j}^{(0)} + \epsilon x_{3j}^{(1)}$ in the boundary value problem, the v-dependence of $x_{31}^{(1)}$ and $x_{32}^{(1)}$ indicates an inconsistency of the simplification of assuming the boundary layer flow as parallel flow. Fortunately, all the previous investigators are satisfied with the first approximation of the two viscous solutions $x_{31}^{(0)}$ and $x_{32}^{(0)}$ while they used $x_{33}^{(0)} + \epsilon x_{33}^{(1)}$ and $x_{34}^{(0)} + \epsilon x_{34}^{(1)}$ or the two equivalent inviscid solutions or the inviscid solutions corrected for viscosity in the boundary value problems. Hence, the stability boundary as determined by any of these methods is independent of v and their results are consistent with the assumption that boundary layer flows are parallel flows. Therefore, within the order of approximation attempted by previous investigators, the stability of the laminar boundary layer is determined only by the *local* flow properties for both the compressible and the incompressible flow.

The accuracy of the quantitative determination of the stability boundary as carried out in references 2, 3 and 4, however, can not be improved merely by taking more terms in the ϵ series without including the effect of the vertical velocity component. It is unfortunate that, in some practical cases, the parameter ϵ may be only 0.1 near the minimum critical Reynolds number and as such the second approximation of the order of ϵ should better be considered for accurate determination of the stability limit and the initial amplification rate, in which cases, the effect of the vertical velocity component must be included.

In addition, at high Mach numbers, the vertical velocity component in the boundary layer is of the order of M^4/R, and may enter the stability problem of the laminar boundary layer even in the first approximation. The stability of the hypersonic laminar boundary layer, therefore, requires careful investigation.

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STRESS-STRAIN RELATIONS, UNIQUENESS AND VARIATIONAL THEOREMS FOR ELASTIC-PLASTIC MATERIALS WITH A SINGULAR YIELD SURFACE*

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1. Plastic stress-strain relations. The state of stress at any point of a continuous medium is described by the stress tensor σ_{ij} and may be represented by a point in nine-dimensional stress space. It will be assumed that no yielding occurs if the stress point

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lies within a convex domain in stress space, which will be called the elastic domain. The boundary of the elastic domain is the yield surface which is called *regular* if it is described by an equation

$$f(\sigma_{ij}) = 0, \tag{1}$$

where f is a regular (continuously differentiable) function, the yield function, of its nine variables, symmetrical with respect to σ_{ij} and σ_{ji} . The sign of f is chosen such that $f(\sigma_{ij})$ is negative in the elastic domain. For a so-called perfectly plastic material that yields under constant stresses, Prager [1] has shown under some very general assumptions that the plastic strain rate tensor is given by

$$\dot{\epsilon}_{ij}'' = \lambda \frac{\partial f}{\partial \sigma_{ij}}, \tag{2}$$

where

$$\left.\begin{array}{llll} \lambda = 0 & \text{for} & f < 0 & \\ & \text{and also for} & f = 0, & \dot{f} \equiv \dfrac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} < 0; \\ \lambda \geq 0 & \text{for} & f = 0, & \dot{f} \equiv \dfrac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = 0. \end{array}\right\} \tag{3}$$

For a material with strain-hardening the relations (2) and (3) are replaced by

$$\left.\begin{array}{llll} \dot{\epsilon}_{ij}'' = 0 & \text{for} & f < 0 & \\ & \text{and also for} & f = 0, & \dot{f} \equiv \dfrac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \leq 0; \\ \dot{\epsilon}_{ij}'' = h \dfrac{\partial f}{\partial \sigma_{ij}} \dot{f} & \text{for} & f = 0, & \dot{f} \equiv \dfrac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \geq 0, \end{array}\right\} \tag{4}$$

where h is a scalar function of stress, plastic strain and strain history. The plastic stress-strain relations (2), (3) and (4) may be called *associated* with the yield function. If the plastic strain rate is depicted in nine-dimensional stress space, these relations are expressed by the geometrical statement that the direction of the plastic strain rate is given by the normal to the yield surface.

In this paper singular yield surfaces will be considered of a type described by a number of regular yield functions $f_\nu(\sigma_{ij})$ (symmetrical with respect to σ_{ij} and σ_{ji}) such that the elastic domain is given by

$$f_\nu(\sigma_{ij}) < 0 \qquad \nu = 1, 2, \cdots n \tag{5}$$

and that yielding occurs as soon as *at least one* of the functions f_ν is zero. All points of the yield surface, where only one function $f_\nu = 0$, are regular and the corresponding plastic strain rates are the same as for a completely regular yield surface $f_\nu = 0$. Difficulties arise only on the intersection of two or more surfaces $f_\nu = 0$ because for these stress states the normal to the yield surface becomes indeterminate. However, this ambiguity is largely removed if the criterion for unloading or yielding at such a singular

point is applied for *each yield function separately*. Physically this assumption is entirely plausible because each yield function represents a *separate* yield criterion. Its mathematical expression is for a perfectly plastic material

$$\dot{\epsilon}_{ij}'' = \sum_{p=1}^{n} \lambda_p \frac{\partial f_p}{\partial \sigma_{ij}}, \tag{6}$$

where

$$\left.\begin{array}{llll} \lambda_p = 0 & \text{for} & f_p < 0, & \\ & \text{and also for} & f_p = 0, & \dot{f}_p \equiv \dfrac{\partial f_p}{\partial \sigma_{ij}} \dot{\sigma}_{ij} < 0; \\ \lambda_p \geq 0 & \text{for} & f_p = 0, & \dot{f}_p \equiv \dfrac{\partial f_p}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = 0 \end{array}\right\} \tag{7}$$

For a material with strain-hardening the relations (6) and (7) are replaced by

$$\dot{\epsilon}_{ij}'' = \sum_{p=1}^{n} c_p h_p \frac{\partial f_p}{\partial \sigma_{ij}} \dot{f}_p, \tag{8}$$

where h_p is a positive function of stress, plastic strain and strain history, and c_p is zero if either $f_p < 0$ or $\dot{f}_p \leq 0$ and unity for $f_p = 0$ and $\dot{f}_p \geq 0$.

2. The variational and uniqueness theorems. Current proofs of the variational and uniqueness theorems for a material with a regular yield surface and the associated plastic stress-strain relations are all of them based on the inequality (see [2])

$$\dot{\sigma}_{ij(1)}\dot{\epsilon}_{ij(1)} + \dot{\sigma}_{ij(2)}\dot{\epsilon}_{ij(2)} - 2\dot{\sigma}_{ij(1)}\dot{\epsilon}_{ij(2)} \geq 0, \tag{9}$$

where $\dot{\sigma}_{ij(1)}$, $\dot{\epsilon}_{ij(1)}$ and $\dot{\sigma}_{ij(2)}$, $\dot{\epsilon}_{ij(2)}$ are two arbitrary pairs of stress rates and associated strain rates at the same point of stress space. The equality sign in (9) holds only if $\dot{\sigma}_{ij(1)} = \dot{\sigma}_{ij(2)}$. This inequality is applied directly to prove the two variational theorems, and the uniqueness theorem is shown to hold by means of the inequality, obtained by interchanging the suffixes 1 and 2 in (9) and adding

$$\dot{\sigma}_{ij(1)}\dot{\epsilon}_{ij(1)} + \dot{\sigma}_{ij(2)}\dot{\epsilon}_{ij(2)} - \dot{\sigma}_{ij(1)}\dot{\epsilon}_{ij(2)} - \dot{\sigma}_{ij(2)}\dot{\epsilon}_{ij(1)} \geq 0. \tag{10}$$

The proof of (9) is obtained by considering the elastic and plastic strain rates separately. Hooke's law leads to the inequality

$$\dot{\sigma}_{ij(1)}\dot{\epsilon}_{ij(1)}' + \dot{\sigma}_{ij(2)}\dot{\epsilon}_{ij(2)}' - 2\dot{\sigma}_{ij(1)}\dot{\epsilon}_{ij(2)}' \geq 0, \tag{11}$$

where $\dot{\epsilon}_{ij}'$ is the elastic strain rate tensor; and the equality sign applies only if $\dot{\sigma}_{ij(1)} = \dot{\sigma}_{ij(2)}$. The similar inequality for the plastic strain rates

$$\dot{\sigma}_{ij(1)}\dot{\epsilon}_{ij(1)}'' + \dot{\sigma}_{ij(2)}\dot{\epsilon}_{ij(2)}'' - 2\dot{\sigma}_{ij(1)}\dot{\epsilon}_{ij(2)}'' \geq 0 \tag{12}$$

is shown to hold by means of the stress-strain relations (2), (3) or (4).

It will now be proved that the stress-strain relations (6)-(8) also lead to inequality (12) and hence ensure the validity of the uniqueness and variational theorems. For a

perfectly plastic material the left-hand member of (12) is calculated as follows

$$\dot\sigma_{ij(1)}\dot\epsilon''_{ij(1)} + \dot\sigma_{ij(2)}\dot\epsilon''_{ij(2)} - 2\dot\sigma_{ij(1)}\dot\epsilon''_{ij(2)}$$
$$= \sum_{p=1}^{n}\left[\lambda_{p(1)}\dot\sigma_{ij(1)}\frac{\partial f_p}{\partial\sigma_{ij}} + \lambda_{p(2)}\dot\sigma_{ij(2)}\frac{\partial f_p}{\partial\sigma_{ij}} - 2\lambda_{p(2)}\dot\sigma_{ij(1)}\frac{\partial f_p}{\partial\sigma_{ij}}\right]$$
$$= \sum_{p=1}^{n}[\lambda_{p(1)}\dot f_{p(1)} + \lambda_{p(2)}\dot f_{p(2)} - 2\lambda_{p(2)}\dot f_{p(1)}].$$

The various possibilities for each term in this sum are listed below with the consequences, ensuing from the stress-strain relations (6), (7):

$$\left.\begin{array}{llll}
f_p < 0 \rightarrow \lambda_{p(1)} = \lambda_{p(2)} = 0, & & & \text{(a)}\\
f_p = 0, & \dot f_{p(1)} < 0, & \dot f_{p(2)} < 0 \rightarrow \lambda_{p(1)} = \lambda_{p(2)} = 0, & \text{(b)}\\
f_p = 0, & \dot f_{p(1)} = 0, & \dot f_{p(2)} < 0 \rightarrow \lambda_{p(1)} \geq 0, \quad \lambda_{p(2)} = 0, & \text{(c)}\\
f_p = 0, & \dot f_{p(1)} < 0, & \dot f_{p(2)} = 0 \rightarrow \lambda_{p(1)} = 0, \quad \lambda_{p(2)} \geq 0, & \text{(d)}\\
f_p = 0, & \dot f_{p(1)} = 0, & \dot f_{p(2)} = 0 \rightarrow \lambda_{p(1)} \geq 0, \quad \lambda_{p(2)} \geq 0, & \text{(e)}
\end{array}\right\} \quad (13)$$

It follows that only if (13d) applies a non-zero term occurs and this term is then always positive. The inequality (12) must therefore hold true and the uniqueness and variational theorems are proved.

The proof of (9) for a strain-hardening material with stress-strain relations (8) is entirely similar and may be omitted here.

3. Application to Tresca's yield criterion for a perfectly plastic material. In some problems, e.g. the thick-walled tube under internal pressure [3], Tresca's yield criterion and its associated stress-strain relations afford a much simpler approach than von Mises' yield condition and its associated Prandtl-Reuss relations. It will now be shown that Tresca's yield criterion belongs to the singular yield criteria considered here.

The most convenient formulation of Tresca's yield criterion for the present purpose is that the shear stress component in any plane, defined by its unit normal $n_i^{(a)}$, and in any direction $n_i^{(b)}$ in this plane (which is of course perpendicular to $n_i^{(a)}$ shall not exceed the yield stress in pure shear. This criterion may be written in a form symmetrical with respect to σ_{ij} and σ_{ji}

$$\frac{1}{2}\sigma_{ij}\{n_i^{(a)}n_j^{(b)} + n_j^{(a)}n_i^{(b)}\} - k \leq 0, \tag{14}$$

where k is the yield stress in shear. Clearly the number of yield functions is now infinite but this does not affect the foregoing argument and the uniqueness and variational theorems remain valid. The plastic strain rates associated with the yield functions (14) follow from (6)

$$\dot\epsilon''_{ij} = \sum_{(a,b)}\frac{1}{2}\lambda_{(ab)}\{n_i^{(a)}n_j^{(b)} + n_j^{(a)}n_i^{(b)}\}, \tag{15}$$

where $\lambda_{(ab)}$ is only non-zero for those directions $n_i^{(a)}$ and $n_i^{(b)}$ for which the equality sign in (14) applies and moreover

$$\dot{\sigma}_{ij}\{n_i^{(a)}n_j^{(b)} + n_j^{(a)}n_i^{(b)}\} = 0. \tag{16}$$

It is obvious that the plastic strain rate, associated with a yield function (14), is a pure shear in the directions $n_i^{(a)}$ and $n_i^{(b)}$.

If the principal stresses are unequal, $\sigma_1 > \sigma_2 > \sigma_3$, there is only one pair of directions $n_i^{(a)}$, $n_i^{(b)}$ for which the equality sign in (14) can hold; this occurs if $\sigma_1 - \sigma_3 = 2\,k$ and the plastic strain rate is a pure shear with its principal axes along the major and minor principal stress axes. If two principal stresses are equal, e.g. $\sigma_1 > \sigma_2 = \sigma_3$, there is a one-parametric family of pairs of directions $n_i^{(a)}$, $n_i^{(b)}$, for which the equality sign in (14) holds. However, if the stress rates are such that $\dot{\sigma}_1 = \dot{\sigma}_3 < \dot{\sigma}_2$, there is only one pair of directions in this family for which (16) holds true and the plastic strain rate is again a pure shear. On the other hand, if both $\sigma_1 - 2k = \sigma_2 = \sigma_3$ and $\dot{\sigma}_1 = \dot{\sigma}_2 = \dot{\sigma}_3$, the plastic strain rate is indeterminate; the stress-strain relations only require that one of the principal axes of the plastic strain rate tensor coincides with the major principal axis of the stress tensor, that the two other principal plastic strain rates are both negative and that the plastic volume strain is zero.

4. The slip theory of Batdorf and Budiansky. Considerable interest has been aroused by the slip theory of plasticity for strain-hardening materials, advanced by Batdorf and Budiansky some years ago [4], partly because it seemed to represent a new approach, entirely different from flow and deformation theories. However, this theory may be regarded as a special case of the present theory for materials with a singular yield surface. The number of regular yield functions, of which the yield condition is composed, is again infinite; the yield functions are similar to (14)

$$\frac{1}{2}\,\sigma_{ij}\{n_i^{(a)}n_j^{(b)} + n_j^{(a)}n_i^{(b)}\} - k_{(ab)} \leq 0, \tag{17}$$

where $k_{(ab)}$ now represents either the initial yield stress in pure shear or (if the latter is higher) the highest previous value of the shear stress component with respect to the two mutually orthogonal directions $n_i^{(a)}$ and $n_i^{(b)}$. The plastic stress-strain relations of Batdorf and Budiansky may be expressed in the form

$$\dot{\epsilon}_{ij}'' = \int c_{(ab)}H_{(ab)}\,\frac{1}{4}\,\{n_i^{(a)}n_j^{(b)} + n_j^{(a)}n_i^{(b)}\}\{n_p^{(a)}n_q^{(b)} + n_q^{(a)}n_p^{(b)}\}\dot{\sigma}_{pq}\,d\Omega_{(ab)}\,, \tag{18}$$

where $d\Omega_{(ab)}$ is the measure of the three-dimensional set of pairs of mutually orthogonal directions $n_i^{(a)}$, $n_i^{(b)}$ within an infinitesimal region, $H_{(ab)}$ is the characteristic strain hardening function of slip theory which depends only on $k_{(ab)}$, and $c_{(ab)}$ plays the role of c_ν in (8). It is easily seen that (18) represents an example of (8) for the case of an infinite number of yield functions f_ν. Consequently the uniqueness and variational theorems are equally applicable to a material that follows the slip theory of plasticity.

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SERIES EXPANSIONS OF RAYS IN ISOTROPIC, NON-HOMOGENEOUS MEDIA*

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The interest in air-to-air radio propagation and its possible use for high-accuracy distance measuring make it useful to have available recursive formulae for the coefficients of the power series expansions of rays. The following note has been extracted from a government report (see Gorn, [6]) with this purpose in mind.

It will be convenient to use a combination of vector and matrix notations. Thus a ray will be given by its position vector: $\mathbf{x} = (x_1\,(t), x_2(t), x_3(t))$. This is a row vector and a 1×3 matrix; the corresponding column vector will be indicated by $\mathbf{x}^T$. Thus the dot product of two vectors is simply $\mathbf{y}\cdot\mathbf{z} = \mathbf{y}\mathbf{z}^T$, while $\mathbf{y}^T\mathbf{z}$ is a 3×3 matrix. The unit 3×3 matrix, with ones down the main diagonal, and zeroes elsewhere, will be represented by I. $\mathbf{y}^2$ will mean $\mathbf{y}\cdot\mathbf{y} = \mathbf{y}\mathbf{y}^T$.

It will also be convenient to use the normalized time parameter

$$u = ct. \tag{1}$$

Here c is the speed of light in a vacuum. Differentiation will be with respect to u, so that, for example, $\mathbf{x}'^T$ will mean the column vector (a 3×1 matrix) whose components are $dx_i/du, i = 1, 2, 3$. Since the index of refraction, $n(x_1, x_2, x_3)$, is defined as c divided by the speed of light in the medium, it follows that along a ray $\mathbf{x}$ we have

$$\mathbf{x}'^2 = n^{-2}. \tag{2}$$

It is at this point that the isotropic property of the medium is used.

Of the various standard forms for the differential equations of rays, which can be found in Herzberger [1], Synge [3], and Luneberg [4], we will therefore use

$$\{n(\mathbf{x}'^2)^{-1/2}\mathbf{x}'\}' = (\mathbf{x}'^2)^{1/2}\nabla n, \tag{3}$$

where ∇n means, of course, the gradient of n:

$$\nabla n = \left(\frac{\partial n}{\partial x_1}, \frac{\partial n}{\partial x_2}, \frac{\partial n}{\partial x_3}\right).$$

Equation (3) is usually simplified by immediate use of equation (2), but new forms of these equations which have direct geometric interpretations can be most readily derived by postponing such use.

First note that from the chain rule for differentiation

$$n' = \mathbf{x}'\cdot\nabla n. \tag{4}$$

Also, differentiating (2) yields

$$\mathbf{x}'\cdot\mathbf{x}'' = -n^{-3}n' = -n^{-3}(\mathbf{x}'\cdot\nabla n). \tag{5}$$

If, then, we perform directly the differentiation of the fraction indicated in the left hand side of (3), and then clear (3) of fractions, a use of (2) and (4), a transposition and a multiplication by n yield

$$n^{-3}\nabla n - n^{-1}(\mathbf{x}'\cdot\nabla n)\mathbf{x}' = \mathbf{x}'' - n^2(\mathbf{x}'\cdot\mathbf{x}'')\mathbf{x}'. \tag{6}$$

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If in (6) we transpose all matrices, we get

$$\{I - n^2\mathbf{x}'^T\mathbf{x}'\}\mathbf{x}''^T = \{I - n^2\mathbf{x}'^T\mathbf{x}'\}(n^{-3}\nabla n)^T. \tag{7}$$

On the other hand we could use (5) either on the left or on the right of (6), transpose a term and then transpose all matrices to obtain the following two forms:

$$\begin{aligned} \mathbf{x}''^T &= \{I - 2n^2\mathbf{x}'^T\mathbf{x}'\}(n^{-3}\nabla n)^T, \\ (n^{-3}\nabla n)^T &= \{I - 2n^2\mathbf{x}'^T\mathbf{x}'\}\mathbf{x}''^T. \end{aligned} \tag{8}$$

It is not difficult to show, as is done in Gorn (6), that $I - 2n^2\mathbf{x}'^T\mathbf{x}'$ is the symmetry operator through the plane perpendicular to $\mathbf{x}'$, while $I - n^2\mathbf{x}'^T\mathbf{x}'$ is the projection operator on to that plane. Equations (8) therefore mean that $\mathbf{x}''$ and $n^{-3}\nabla n$ are symmetric images through the plane perpendicular to $\mathbf{x}'$, while equation (7) states that they have the same projection on to that plane. The three component equations of (7) must, consequently, be dependent, since many vectors have the same projection as $n^{-3}\nabla n$; (7) contains incomplete information about $\mathbf{x}''$ and needs, say, (2) as an adjunct. Either set of (8), on the other hand, contains complete information.

At this point we can make use of the standard differential geometry of space curves (see e.g. Franklin [5], p. 107 on) with its concepts of curvature κ, torsion τ, the natural coordinate vectors, namely the unit tangent $\mathbf{T}$, the unit normal $\mathbf{N}$, and the unit binormal $\mathbf{B}$, and finally the Frenet-Serret formulae expressing the derivatives of the natural coordinate vectors as linear combinations of $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$. Once we find the curvature and torsion along rays we will be able to differentiate any vector field defined along them by using the Frenet-Serret formulae. In particular, repeated differentiation of $\mathbf{x}$ by this means will yield power series expansions of rays. These expansions correspond to the one well-known in differential geometry for space curves.

If s is the arc length measured along a ray, then $s' = n^{-1}$. Furthermore, the transpose of the first form of (8) is

$$\mathbf{x}'' = -2n^{-1}(\mathbf{x}'\cdot\nabla n)\mathbf{x}' + n^{-3}\nabla n. \tag{9}$$

From these and the curvature formula, one readily derives the well-known formulae for the curvature of a ray:

$$\kappa = |\,\mathbf{x}' \times \nabla n\,| = -n^{-1}\,|\,\nabla n\,|\sin\psi, \tag{10}$$

where ψ is the angle between $\mathbf{x}'$ and ∇n.

The standard forms for $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$, with the help of equations (5) and (9) become

$$\begin{aligned} \mathbf{T} &= n\mathbf{x}' \\ \mathbf{N} &= -n\kappa^{-1}(\mathbf{x}'\cdot\nabla n)\mathbf{x}' + (n\kappa)^{-1}\nabla n \\ \mathbf{B} &= \kappa^{-1}\mathbf{x}' \times \nabla n. \end{aligned} \tag{11}$$

To find a formula for the torsion of a ray, one differentiates (9), and uses the rules or simplifying a determinant to obtain the scalar triple product

$$(\mathbf{x}'\mathbf{x}''\mathbf{x}''') = n^{-6}(\mathbf{x}'\ \nabla n\ [\nabla n]').$$

Since

$$[\nabla n]' = \mathbf{x}'\ \nabla_2 n, \tag{12}$$

where

$$\nabla_2 n = \left(\frac{\partial^2 n}{\partial x_i\, \partial x_j}\right), \tag{13}$$

a symmetric matrix, the standard torsion formula yields

$$\tau = \kappa^{-2}(\mathbf{x}'\ \nabla n\ [\nabla n]') = \kappa^{-2}(\mathbf{x}'\ \nabla n\ \mathbf{x}'\ \nabla_2 n). \tag{14}$$

Alternative forms for κ and τ can be given in terms of the matrix

$$Y = \begin{bmatrix} 0 & -\dfrac{\partial n}{\partial x_3} & \dfrac{\partial n}{\partial x_2} \\[2ex] \dfrac{\partial n}{\partial x_3} & 0 & -\dfrac{\partial n}{\partial x_1} \\[2ex] -\dfrac{\partial n}{\partial x_2} & \dfrac{\partial n}{\partial x_1} & 0 \end{bmatrix}$$

which is skew symmetric: $Y^T = -Y$, and has the further properties that

$$Y^2 = (\nabla n)^T\ \nabla n - (\nabla n)^2 I,$$
$$\mathbf{w}Y = \mathbf{w} \times \nabla n$$

for any row vector $\mathbf{w}$, and has the characteristic polynomial $\lambda\{\lambda - (\nabla n)^2\}^2$.

$$\kappa^2 = -\mathbf{x}' Y^2 \mathbf{x}'^T,$$
$$\kappa^2 \tau = -\mathbf{x}' Y(\nabla_2 n)\mathbf{x}'^T.$$

The knowledge of κ and τ permits us to express the Frenet-Serret formulae for rays

$$\begin{aligned} \mathbf{T}' &= & \kappa n^{-1}\mathbf{N} & \\ \mathbf{N}' &= -\kappa n^{-1}\mathbf{T} & & +\tau n^{-1}\mathbf{B} \\ \mathbf{B}' &= & -\tau n^{-1}\mathbf{N} & \end{aligned} \tag{15}$$

We now have all the apparatus needed to obtain recursion formulae for the expression of the m'th derivative of any vector $\mathbf{y}$ if one knows the coefficients in $\mathbf{y} = a_0\mathbf{T} + b_0\mathbf{N} + c_0\mathbf{B}$ or $\mathbf{y} = d_0\mathbf{x}' + e_0\ \nabla n + f_0(\mathbf{x}' \times \nabla n)$; let us begin with the first form.

If we have

$$\mathbf{y}^{(m)} = a_m\mathbf{T} + b_m\mathbf{N} + c_m\mathbf{B}, \tag{16}$$

then differentiation and application of Frenet's formulae (15) yield

$$\mathbf{y}^{(m+1)} = \{a_m' - \kappa n^{-1} b_m\}\mathbf{T} + \{b_m' + \kappa n^{-1} a_m - \tau n^{-1} c_m\}\mathbf{N} + \{c_m' + \tau n^{-1} b_m\}\mathbf{B}.$$

Therefore:

$$\begin{aligned} a_{m+1} &= a_m' & - \kappa n^{-1} b_m & & , \\ b_{m+1} &= b_m' + \kappa n^{-1} a_m & & - \tau n^{-1} c_m & , \\ c_{m+1} &= c_m' & + \tau n^{-1} b_m & & . \end{aligned} \tag{17}$$

As an example, the first equation of (11) now reads

$$\text{for} \quad \mathbf{y} = \mathbf{x}, \quad \begin{aligned} a_1 &= n^{-1}, \\ b_1 &= 0, \\ c_1 &= 0. \end{aligned} \tag{18}$$

Applying (17) therefore yields

$$\text{for} \quad \mathbf{y} = \mathbf{x}, \quad \begin{aligned} a_2 &= -n'n^{-2}, \\ b_2 &= \kappa n^{-2}, \\ c_2 &= 0. \end{aligned} \tag{19}$$

This checks with equation (9) in view of (4) and the second equation of (11).

Similarly, eliminating $\mathbf{x}'$ from the first two equations of (11) and solving for ∇n yields

$$\text{for} \quad \mathbf{y} = \nabla n, \quad \begin{aligned} a_0 &= nn', \\ b_0 &= n\kappa, \\ c_0 &= 0. \end{aligned} \tag{20}$$

Here again, then, we can apply recursion formulae (17) to give

$$\text{for} \quad \mathbf{y} = \nabla n, \quad \begin{aligned} a_1 &= [nn']' - \kappa^2, \\ b_1 &= [n\kappa]' + \kappa n', \\ c_1 &= \kappa\tau. \end{aligned} \tag{21}$$

This formula for $[\nabla n]' = \mathbf{x}' \nabla_2 n$ can be given an alternative form, for total derivatives of n and κ can always be reduced to expressions involving partial derivatives of n and κ with respect to the x_i and total derivatives in $\mathbf{x}$. For example, a_1 is the $\mathbf{T}$ component of $[\nabla n]' = \mathbf{x}' \nabla_2 n$, hence $a_1 = \mathbf{x}' \nabla_2 n \cdot \mathbf{T} = n\mathbf{x}' \nabla_2 n \mathbf{x}'^T$.

For the alternative form of b_1, we need such an expression for κ', which, in turn, needs one for n''. From the Lagrange identity and formula (10),

$$\kappa^2 = |\,\mathbf{x}' \times \nabla n\,|^2 = n^{-2}(\nabla n)^2 - n'^2. \tag{22}$$

From this and the alternative expression for a_1 it is easy to show that

$$n'' = \mathbf{x}' \nabla_2 n\, \mathbf{x}'^T + n^{-3}(\nabla n)^2 - 2n^{-1}n'^2. \tag{23}$$

Finally, from (23), (12), and a differentiation of (22), we find

$$\kappa' = -2n^{-1}n'\kappa - n'\kappa^{-1}(\mathbf{x}' \nabla_2 n\, \mathbf{x}'^T) + n^{-2}\kappa^{-1}(\nabla n\, \nabla_2 n\, \mathbf{x}'^T). \tag{24}$$

Thus (21) may also be written as follows:

$$\text{for} \quad \mathbf{y} = \nabla n, \quad \begin{aligned} a_1 &= n(\mathbf{x}' \nabla_2 n\, \mathbf{x}'^T), \\ b_1 &= -nn'\kappa^{-1}(\mathbf{x}' \nabla_2 n\, \mathbf{x}'^T) + n^{-1}\kappa^{-1}(\nabla n\, \nabla_2 n\, \mathbf{x}'^T), \\ c_1 &= \kappa\tau. \end{aligned} \tag{21a}$$

This form for $[\nabla n]'$ permits us to return to the computation of $\mathbf{x}'''$ from (19) and (17). Naturally the same result could be obtained by using (23) and (24). In either case a computation yields

$$\text{for}\quad \mathbf{y} = \mathbf{x}, \qquad \begin{aligned} a_3 &= -n^{-3}\{2\kappa^2 - 3n'^2 + n(\mathbf{x}' \nabla_2 n\, \mathbf{x}'^T)\}, \\ b_3 &= -n^{-3}\{5n'\kappa + \kappa^{-1}nn'(\mathbf{x}' \nabla_2 n\, \mathbf{x}'^T) - n^{-1}\kappa^{-1}(\nabla n \nabla_2 n\, \mathbf{x}'^T)\}, \\ c_3 &= n^{-3}\kappa\tau. \end{aligned} \tag{25}$$

Further application of (17) to $\mathbf{y} = \mathbf{x}$ is complicated by the fact that higher order derivatives of n are involved beyond n'', i.e. beyond those in ∇n and $\nabla_2 n$; they are probably best handled by introducing multiple index symbols, such as are used in tensor calculus. At this point the matrix notation stops being convenient. So much for the recursion formulae of type (16). Now let us consider those of type

$$\mathbf{y}^{(m)} = d_m\mathbf{x}' + e_m \nabla n + f_m(\mathbf{x}' \times \nabla n). \tag{26}$$

To avoid confusion let us designate the d_1, e_1, and f_1 when $\mathbf{y} = \nabla n$ by p_1, q_1, r_1; if we apply (11) to (21a) we obtain

$$[\nabla n]' = p_1\mathbf{x}' + q_1 \nabla n + r_1(\mathbf{x}' \times \nabla n),$$

where

$$\begin{aligned} p_1 &= \kappa^{-2}\{(\nabla n)^2(\mathbf{x}' \nabla_2 n\, \mathbf{x}'^T) - n'(\nabla n \nabla_2 n\, \mathbf{x}'^T)\}, \\ q_1 &= \kappa^{-2}\{-n'(\mathbf{x}' \nabla_2 n\, \mathbf{x}'^T) + n^{-2}(\nabla n \nabla_2 n\, \mathbf{x}'^T)\}, \\ r_1 &= \tau. \end{aligned} \tag{27}$$

Differentiating (26) and using (9) with (27) yields

$$\begin{aligned} d_{m+1} &= d_m' - 2n^{-1}n'd_m + p_1e_m + n'r_1f_m, \\ e_{m+1} &= e_m' + n^{-3}d_m \quad + q_1e_m - n^{-2}r_1f_m, \\ f_{m+1} &= f_m' \qquad\qquad\quad + r_1e_m + \{q_1 - 2n^{-1}n'\}f_m. \end{aligned} \tag{28}$$

As examples, if $\mathbf{y} = \mathbf{x}$ in (26) and $m = 1$, then obviously $d_1 = 1$, $e_1 = 0$, $f_1 = 0$; hence, from (28), $d_2 = -2n^{-1}n'$, $e_2 = n^{-3}$, $f_2 = 0$, which is the same as (9). Another application of (28) with $m = 2$ yields

$$\text{for}\quad \mathbf{y} = \mathbf{x}, \qquad \begin{aligned} d_3 &= -2(n^{-1}n')' + 4n^{-2}n'^2 + n^{-3}p_1, \\ e_3 &= (n^{-3})' - 2n^{-4}n' \qquad + n^{-3}q_1, \\ f_3 &= \qquad\qquad\qquad\qquad\quad n^{-3}r_1. \end{aligned} \tag{29}$$

Again, if $\mathbf{y} = \nabla n$, then $d_0 = 0$, $e_0 = 1$, $f_0 = 0$, hence $d_1 = p_1$, $e_1 = q_1$, $f_1 = r_1$ as in (27).

To obtain power series expansions of the rays

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_0'u + \frac{1}{2}\mathbf{x}_0''u^2 + \cdots + \frac{1}{m!}\mathbf{x}_0^{(m)}u^m + \cdots, \tag{30}$$

one can either apply (16) to $\mathbf{y} = \mathbf{x}$ at $u = 0$ (the first four terms come from (18), (19), and (25) if one desires to use **T**, **N**, and **B** as coordinate vectors), or one can apply (26) and use the recursion formulae (28), as just indicated.

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ON SUPERSONIC FLOW PAST AN OSCILLATING WEDGE*

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1. Introduction. In order to study the limitations of the linearized theory of oscillating airfoils, Carrier[1] has analyzed supersonic flow past a thick wedge which oscillates slightly about its apex. Considerably greater insight into the nonlinear effects of thickness is gained by generalizing the analysis to include other locations of the pivot.

2. Analysis. Consider a wedge fixed in a supersonic stream at a Mach number above that at which the shock wave detaches from the apex. If now the wedge executes slight oscillations, the shock will remain attached, and the flow field can be found by superposing a linearized acoustic field upon the nonlinear steady flow. For harmonic oscillations of frequency a_2c, Carrier has shown [1,2] that the velocity components and pressure downstream of the shock are given by

$$u = U_2 + a_2(\varphi_x + E_y), \qquad v = a_2(\varphi_y - E_x), \tag{1}$$

$$p = p_2[1 - \gamma(\varphi_t + M\varphi_x)], \tag{2}$$

where

$$\varphi = e^{ic(t - Mx/\beta^2)} \sum^{\infty} (a_\nu \cosh \nu\theta + b_\nu \sinh \nu\theta) J_\nu(kr), \tag{3}$$

$$E = e^{ic(t - x/M - \lambda y/M\beta^2)} \sum^{\infty} c_\nu J_\nu(k\xi y), \tag{4}$$

and the deflection of the shock wave from its steady position is

$$\psi = (1 - \rho_1/\rho_2)^{-1} e^{ic(t - \lambda My/\beta^2)} \sum^{\infty} d_\nu J_\nu(k\xi y). \tag{5}$$

Here, U_2 , a_2 , M, p_2 and ρ_2 are the flow speed, sound speed, Mach number, pressure and density downstream of the shock in steady flow (and ρ_1 the density upstream), γ is the adiabatic exponent, t the time multiplied by a_2 , $\beta^2 = M^2 - 1$, $k = c/\beta^2$, $\tanh \theta = \beta y/x$, $r^2 = x^2 - \beta^2 y^2$ and $\xi^2 = 1 + \lambda^2 - M^2$. The coordinate system and λ are defined in Fig. 1.

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[1]G. F. Carrier, *The oscillating wedge in a supersonic stream*, J. Aer. Sci., **16**, 150-152 (1949).

[2]G. F. Carrier, *On the stability of the supersonic flows past a wedge*, Q. Appl. Math. **6**, 356-378 (1949).

The pertinent jump conditions across the shock wave, which can be imposed at the steady shock position, were shown to be equivalent to

$$\varphi_x + E_y = (1 - \rho_1/\rho_2)[(n\alpha_1 + \lambda\alpha_2)\psi_y/(1 + \lambda^2) + \alpha_1\psi_t/(1 + \lambda^2)^{1/2}], \tag{6}$$

$$\varphi_y - E_x = (1 - \rho_1/\rho_2)[(\alpha_2 - \lambda n\alpha_1)\psi_y/(1 + \lambda^2) - \lambda\alpha_1\psi_t/(1 + \lambda^2)^{1/2}], \tag{7}$$

$$\varphi_t + M\varphi_x = -(1 - \rho_1/\rho_2)\alpha_3[n\psi_y/(1 + \lambda^2)^{1/2} + \psi_t], \tag{8}$$

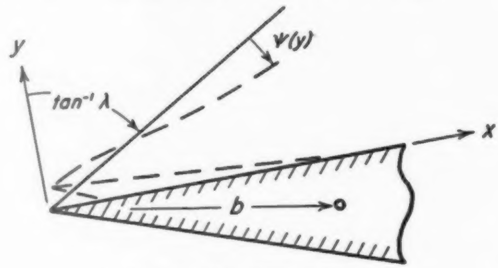

FIG. 1. Coordinate system.

where $\alpha_1 = (2 + 4m^2)/3(1 - m^2)$, $\alpha_2 = (5 + m^2)/6m$ and $\alpha_3 = -m(5 + m^2)/3(1 - m^2)$ for $\gamma = 7/5$. Here m and n are the components of M normal and tangential to the undisturbed shock.

Suppose now that the wedge is pivoted at a point a distance b downstream of its apex, and oscillates with small angle of attack $\alpha = Re(e^{ict})$ in some unit system. The boundary condition of tangent flow at the surface is[3]

$$(\varphi_y - E_x)_{y=0} = -[M + ic(x - b\cos\epsilon)]e^{ict}, \tag{9}$$

where ϵ is the semi-vertex angle of the wedge. Following Carrier, using the generating function for the Bessel coefficients we find that for $\nu \geq 1$ the b_ν are given by

$$b_\nu = i\nu[\tau^\nu + (-\tau)^{-\nu}]/\beta k + b\cos\epsilon[\tau^\nu - (-\tau)^{-\nu}], \tag{10}$$

where $\tau = i(M + \beta)$.

For the special case of rotation about the apex, the summations in Eqs. (3) to (5) begin with $\nu = 1$. For other pivot positions, however, a term $\nu = 0$ must be added to Eq. (5) to account for the fact that as the apex of the wedge oscillates about the origin the attached shock wave moves with it. The question of whether corresponding modifications are required in the series for φ and E can be answered as follows. Consider the pivot point to be moved indefinitely far downstream, and the amplitude of angular oscillation correspondingly reduced, so that ultimately the wedge simply executes a small vertical translational ("plunging") oscillation. Furthermore, let the frequency of oscillation tend to zero. In the limit, the wedge stands fixed and slightly above and parallel with its original position. It is clear that in this steady flow the shock wave is displaced from its original position, but the velocity perturbations associated with φ and E are zero. Hence a $\nu = 0$ mode must be added only to Eq. (5).

[3]Here an error in the tangency condition in the reference of footnote 1 has been corrected. Furthermore, numerous typographical errors in the subsequent equations of that paper have been rectified.

The additional coefficient thus introduced is determined by the condition that the shock wave must at all times meet the apex of the wedge, which gives

$$d_0 = -b(1 - \rho_1/\rho_2)(1 + \lambda^2)^{-1/2}(\lambda \cos \epsilon - \sin \epsilon). \tag{11}$$

Then the shock wave conditions yield three recurrence relations for determining successively the coefficients a_ν, c_ν and $d_\nu (\nu \geq 1)$ in terms of d_0 and the b_ν. In matrix form, with $\tanh \theta_0 = \beta/\lambda$, these are

$$\begin{bmatrix} \cosh \nu\theta_0 & \xi & -n\sqrt{1-m^2}\,(m\alpha_1 + \alpha_2)/M \\ \beta \sinh \nu\theta_0 & 0 & \sqrt{1-m^2}\,(n^2\alpha_1 - m\alpha_2)/M \\ -M \cosh \nu\theta_0 & 0 & -n\sqrt{1-m^2}\,\alpha_3 \end{bmatrix} \begin{bmatrix} a_{\nu+1} \\ c_{\nu+1} \\ d_{\nu+1} \end{bmatrix}$$

$$+ \begin{bmatrix} -\cosh \nu\theta_0 & -\xi & n\sqrt{1-m^2}\,(m\alpha_1 + \alpha_2)/M \\ \beta \sinh \nu\theta_0 & 0 & -\sqrt{1-m^2}\,(n^2\alpha_1 - m\alpha_2)/M \\ M \cosh \nu\theta_0 & 0 & n\sqrt{1-m^2}\,\alpha_3 \end{bmatrix} \begin{bmatrix} a_{\nu-1} \\ c_{\nu-1} \\ d_{\nu-1} \text{ but } 2d_0 \end{bmatrix}$$

$$+ \begin{bmatrix} -2iM \cosh \nu\theta_0 & -2i\lambda/M & 2i[m(1-m^2)\alpha_1 + n^2\alpha_2]/M \\ 0 & 2i\beta^2/M & -2in[(1-m^2)\alpha_1 - m\alpha_2]/M \\ 2i \cosh \nu\theta_0 & 0 & 2i(1-m^2)\alpha_3 \end{bmatrix} \begin{bmatrix} a_\nu \\ c_\nu \\ d_\nu \end{bmatrix}$$

$$+ \begin{bmatrix} \sinh \nu\theta_0 & -\sinh \nu\theta_0 & -2iM \sinh \nu\theta_0 \\ \beta \cosh \nu\theta_0 & \beta \cosh \nu\theta_0 & 0 \\ -M \sinh \nu\theta_0 & M \sinh \nu\theta_0 & 2i \sinh \nu\theta_0 \end{bmatrix} \begin{bmatrix} b_{\nu+1} \\ b_{\nu-1} \\ b_\nu \end{bmatrix} = 0, \tag{12}$$

for $\nu = 0, 1 \cdots$, where it is understood that the a_ν, b_ν and c_ν vanish for $\nu < 1$, and the d_ν for $\nu < 0$. Note that for $\nu = 1$, $d_{\nu-1}$ is to be replaced by $2d_0$.

3. Example: Slow oscillations. The solution can be readily converted into an expansion in powers of frequency. Then retaining only linear terms in frequency shows that for slow oscillations the surface pressure coefficient, referred to conditions upstream of the shock, is

$$C_p \equiv (p - p_1)/\tfrac{1}{2}\rho_1 U_1^2 = \overline{C}_p - (2/M_1^2 + \gamma \overline{C}_p)[A\alpha + (Bb \cos \epsilon + Cx)\dot{\alpha}/a_2]. \tag{13}$$

Here $\overline{C}_p$ is the value for steady flow, M_1 the free-stream Mach number, and α and $\dot{\alpha}$ the instantaneous angle of attack and its (true) time derivative. The coefficients A, B, C are

$$A = -M^2 n\alpha_3/\mu \qquad B = M(n - m \tan \epsilon)\alpha_3/\mu,$$

$$C = M \frac{2\mu - 2M^2\lambda n\alpha_3/\beta^2 + [(M^2 + n^2)\alpha_2 - mn^2]\alpha_3/\mu}{\lambda\mu - \beta^2 n\alpha_3} + \frac{M(M^2 + 1)n\alpha_3}{\beta^2\mu}, \tag{14}$$

where $\mu = n^2\alpha_1 - m\alpha_2$. For a wedge airfoil with flat base (on which the unknown base pressure is assumed to be constant, though possibly time dependent) the normal force and pitching moment coefficients are

$$C_N = (2/M_1^2 + \gamma\overline{C}_p)[2A\alpha + (2B\sigma + C)\alpha^{\cdot} \sec \epsilon/a_2], \tag{15}$$

$$C_M = (2/M_1^2 + \gamma\overline{C}_p) \sec^2 \epsilon[A(2\sigma - 1)\alpha + \{B\sigma(2\sigma - 1) + C(\sigma - \tfrac{2}{3})\}\alpha^{\cdot} \sec \epsilon/a_2], \tag{16}$$

where $\sigma = c \cos^2 \epsilon/c$.

The term in Eq. (16) proportional to $\alpha^{\cdot}$ represents the aerodynamic damping moment for slow oscillations, which tends to stabilize if it is negative. The boundary of neutral stability as it depends upon pivot position and free-stream Mach number is shown in Fig. 2 for a wedge airfoil of 5° semi-vertex angle. Also shown are the corresponding

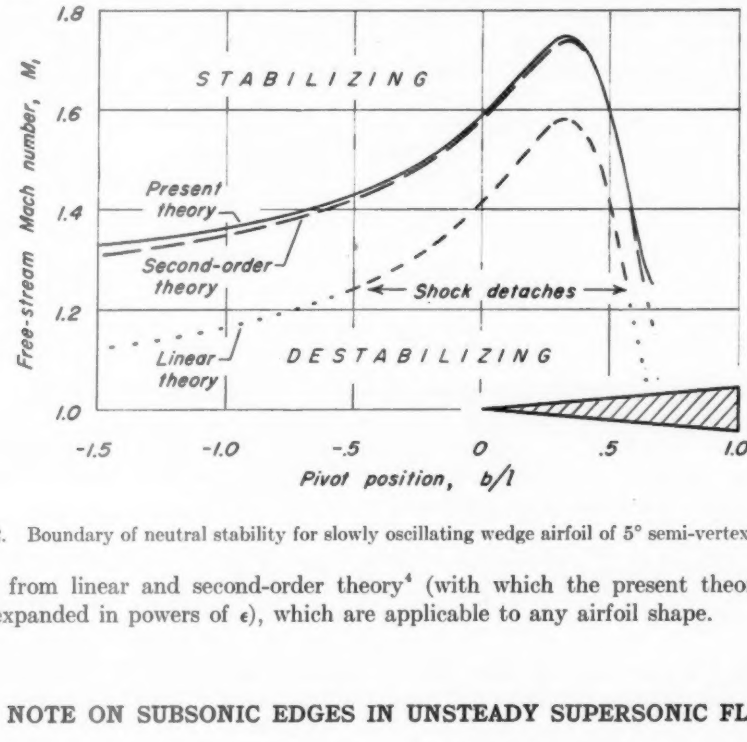

FIG. 2. Boundary of neutral stability for slowly oscillating wedge airfoil of 5° semi-vertex angle.

results from linear and second-order theory[4] (with which the present theory agrees when expanded in powers of ϵ), which are applicable to any airfoil shape.

A NOTE ON SUBSONIC EDGES IN UNSTEADY SUPERSONIC FLOW*

BY JOHN W. MILES (*University of California, Los Angeles*)

Summary. The pressure distribution due to the unsteady motion of a wing having a supersonic leading edge and a subsonic trailing edge is determined by applying a Lorentz transformation to the corresponding result for a rectangular wing. This result, valid as

[4]Milton D. Van Dyke, *On second-order supersonic flow past a slowly oscillating airfoil*, J. Aero. Sci. 20, 61 (1953).

*Received March 2, 1953.

it stands for a subsonic leading edge, is then modified by the addition of an eigensolution in order to satisfy the Kutta condition.

1. Introduction. It was noted in an earlier paper [1] that the general solution for a rectangular wing in unsteady supersonic flow could be extended to a wing having a straight subsonic leading edge with the aid of the Lorentz transformation. It is the purpose of the present note to extend this result to a subsonic trailing edge. (The corresponding steady flow problem has been treated by Lagerstrom [2] and others.)

2. The potential equation. The notation of the present paper is in agreement with that of ref. 1, to which we refer by the prefix I ( ), where ( ) denotes the equation number therein. In the interests of brevity, the following analysis is written as a continuation of I and repeats equations and results stated therein only insofar as appears necessary to define the problem at hand.

The original coordinates (dimensionless, being referred to a characteristic length l) are the orthogonal Cartesian set (x, y, z), where the free stream velocity U, is directed along $+x$. In these coordinates, the projection on $z = 0$ of the wing to be considered is bounded by

$$x + m\beta y > 0 \tag{2.1a}$$

$$m\beta x + y \geqq 0 \tag{2.1b}$$

$$-1 < m \leqq 0 \tag{2.2}$$

$$\beta = (M^2 - 1)^{1/2} \tag{2.3}$$

where $x + m\beta y = 0$ defines a straight, supersonic leading edge and $m\beta x + y = 0$ a straight, subsonic trailing edge. The definition of the leading edge is dictated by convenience (so $x' = 0$ there; *v.i.*) and may be varied at will, insofar as it remains supersonic.

We next introduce the coordinates (ξ, τ) by cascading Gallilean [I (2.3)] and modified Lorentz [I(2.4)] transformations as follows:

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} = \beta^{-1}\begin{pmatrix} 1 & M \\ M & 1 \end{pmatrix}\begin{pmatrix} 1 & -M \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ at/l \end{pmatrix} \tag{2.4}$$

where t and a represent true time and the sonic velocity, respectively. Finally, we introduce the oblique coordinates (x', y') *via* the true Lorentz [I(8.1)] transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (1 - m^2)^{-1/2}\begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}\begin{pmatrix} \xi \\ y \end{pmatrix} \tag{2.5}$$

In these coordinates the leading and trailing edges are specified by $x' = 0$ and $y' = 0$, respectively. Then, taking (x', y', z, τ) as our working coordinates, the equation for the velocity potential, obtained by applying the Lorentz transformation to I(2.6), reads

$$\phi_{x'x'} - \phi_{y'y'} - \phi_{zz} = \phi_{\tau\tau} \tag{2.6}$$

The corresponding equation for the pressure perturbation, obtained by applying (2.5) to I(2.12), may be written

$$p = -\rho_0 a \beta^{-1} \chi \tag{2.7}$$

$$\chi = \chi(x', y', z, \tau) = (1 - m^2)^{-1/2} M(\phi_{x'} + m\phi_{y'}) + \phi_\tau \tag{2.8}$$

The integration of (2.8), subject to the boundary condition $\phi = 0$ at the leading edge, yields,

$$\phi = (1 - m^2)^{1/2} M^{-1} \int_0^{x'} \chi[\mu, y' - m(x' - \mu), \tau - (1 - m^2)^{1/2} M^{-1}(x' - \mu)] \, d\mu \tag{2.9}$$

3. The boundary conditions. The wing platform (**S**), as it appears in the (ξ, y) and (x', y') coordinates, is shown in Fig. 1. The wake aft of the trailing edge is designated

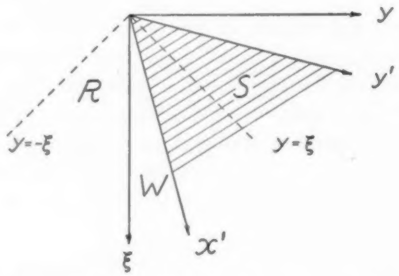

FIG. 1. The regions **S**, **W** and **R**, the various coordinate axes, and the Mach lines $y = \pm\xi$.

as **W** and the remainder of the plane $z = 0$ as **R**. The boundary conditions in **S** and **R** are as in I(2.7) and I(2.10), respectively, while the boundary condition in **W** is dictated by the requirement of continuity of pressure. Thus, we obtain

$$\phi_z = -w'(x', y', \tau) \quad \text{on} \quad \mathbf{S}(x' > 0, y' > 0, z = 0\pm) \tag{3.1}$$

$$\chi = 0 \quad \text{on} \quad \mathbf{W}(x' > 0, mx' \leqq y' \leqq 0, z = 0) \tag{3.2}$$

$$\phi = 0 \quad \text{on} \quad \mathbf{R}(x' > 0, y' < mx', z = 0) \tag{3.3}$$

where w', the prescribed downwash, is now exhibited as a function of the oblique coordinates.

In the case of a non-trailing edge the region **W** would not exist, and (3.3) would hold for $y' \leqq 0$. The corresponding solution for ϕ, designated as $\phi^{(1)}$ in the following, and its derivatives $\phi_{x'}$, and ϕ_τ then would vanish as $y'^{+1/2}$ at the edge $y' = 0$, while $\phi_{y'}$, would behave as $y'^{-1/2}$ there. Accordingly, for $m > 0$ χ would exhibit the expected singularity associated with a subsonic leading edge (*cf.* ref. 2). On the other hand, for $m < 0$ an eigensolution, $\phi^{(2)}$, can be determined to cancel this singularity, thereby satisfying the Kutta condition ($\chi = 0$ at $y' = 0$). Accordingly, we write

$$\phi = \phi^{(1)} + \phi^{(2)} \tag{3.4}$$

and formulate two, subsidiary boundary value problems.

As indicated in the foregoing, $\phi^{(1)}$ may be assumed to satisfy the boundary conditions

$$\phi_z^{(1)} = -w' \qquad \text{on} \qquad \mathbf{S}(x' > 0,\, y' > 0,\, z = 0\pm) \tag{3.5}$$

$$\phi^{(1)} = 0 \qquad \text{on} \qquad \mathbf{W} + \mathbf{R}(x' > 0,\, y' \leqq 0,\, z = 0) \tag{3.6}$$

It follows that $\phi_z^{(2)}$ and, therefore, $\chi_z^{(2)}$ must vanish in **S** in order to satisfy (3.1). Moreover, since $\phi^{(2)}$ must vanish in **R**, so also must $\chi^{(2)}$. Hence, we write

$$\chi_z^{(2)} = 0 \qquad \text{on} \qquad \mathbf{S} \tag{3.7}$$

$$\chi^{(2)} = 0 \qquad \text{on} \qquad \mathbf{W} + \mathbf{R} \tag{3.8}$$

To complete the formulation, we remark that χ also satisfies the differential equation (2.6).

As will be shown below, the explicit determination of $\phi^{(2)}$ is not required insofar as χ alone is required. However, given $\chi^{(2)}$, $\phi^{(2)}$ may be determined from (2.9).

4. Solution. The boundary value problem of determining $\phi^{(1)}$ when w' is prescribed is now identical with the rectangular wing problem solved in I. A convenient form of the solution for our present purpose is obtained by posing the harmonic time dependence $\exp(i\kappa\tau)$, substituting x' and y' for ξ and y in I(4.5) and introducing the trigonometric change of variable $\eta = y' + (x' - \mu) \cos\theta \cos\lambda$, whence we have

$$\begin{aligned} \phi^{(1)}(x', y', 0+, \tau) = {} & \int_0^{x'} J_0[\kappa(x' - \mu)] w'(\mu, y', \tau)\, d\mu \\ & - \frac{1}{\pi} \int_0^{x'} d\mu \int_0^{\pi/2} J_0[\kappa(x' - \mu) \sin\theta]\, d\theta \\ & \cdot \frac{\partial}{\partial\theta} \int_0^{2\sin^{-1}[y'/(x'-\mu)\cos\theta]^{1/2}} w'[\mu, y' + (x' - \mu)\cos\theta\cos\lambda, \tau]\, d\lambda \end{aligned} \tag{4.1}$$

where the arc-sine is to be replaced by $\pi/2$ when its argument exceeds unity. In the following, we refer to this solution in the symbolic form

$$\phi^{(1)}(x', y', 0+, \tau) = \phi^{(1)}\{w'(x', y', \tau)\} \tag{4.2}$$

To obtain $\chi^{(2)}$, we remark that, since any derivative of ϕ also satisfies (2.6), $\phi_{y'}^{(1)}\{w'\}$ represents a solution satisfying the boundary conditions (3.5) and (3.6) provided that w' is replaced by $w'_{y'}$ therein. Moreover, $\phi^{(1)}\{w'_{y'}\}$ satisfies the same differential equation and boundary condition as $\phi_{y'}^{(1)}$ but differs therefrom in vanishing at $y' = 0$, whereas $\phi_{y'}^{(1)}$ is singular as $y'^{-1/2}$ in this neighborhood. It follows that $\phi_{y'}^{(1)}\{w'\} - \phi^{(1)}\{w'_{y'}\}$ represents a solution to the differential equation (2.6) satisfying the homogeneous boundary conditions (3.7) and (3.8) and exhibiting the desired singularity at $y' = 0$. Accordingly, the required solution for $\chi^{(2)}$ that cancels the singularity in $\chi^{(1)}$ at $y' = 0$ is given by

$$\chi^{(2)} = -m(1 - m^2)^{-1/2} M[\phi_{y'}^{(1)}\{w'\} - \phi^{(1)}\{w'_{y'}\}] \tag{4.3}$$

Substituting from (4.1), we have

$$\chi^{(2)} = m(1 - m^2)^{-1/2} M y'^{-1/2} \pi^{-1} \int_0^{x'} d\mu \int_0^{\cos^{-1}[y'/(x'-\mu)]} J_0[\kappa(x' - \mu) \sin \theta)] \cdot \frac{\partial}{\partial \theta} \{[(x' - \mu) \cos \theta - y']^{-1/2} w'[\mu, (x' - \mu) \cos \theta - y', \tau]\} \, d\theta \tag{4.4}$$

The final solution for the pressure distribution on the upper surface of the wing, obtained by substituting $\phi^{(1)}$ in (2.8) and adding $\chi^{(2)}$ from (4.3), is given by

$$\chi(x', y', 0+, \tau) = (1 - m^2)^{-1/2} M[\phi_{x'}^{(1)}\{w'\} + m\phi^{(1)}\{w'_{y'}\}] + \phi_\tau^{(1)}\{w'\} \tag{4.5}$$

We remark that the results (4.3) and (4.5) are not restricted to harmonic time dependence, since they are valid for all frequencies.

In the case of a subsonic leading edge ($m \geq 0$), it would be necessary only to replace $\phi^{(1)}\{w'_{y'}\}$ by $\phi_{y'}^{(1)}\{w'\}$ in (4.3).

REFERENCES

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2. P. A. Lagerstrom, *Linearized supersonic theory of conical wings*, J. P. L. Prog. Rep. 4-36 Pasadena, (1947); reprinted as NACA TN 1685 (1948), 90-98.

NOTE ON THE MEAN SQUARE VALUE OF INTEGRALS IN THE STATISTICAL THEORY OF TURBULENCE*

By C. C. LIN (*Massachusetts Institute of Technology*)

1. In the statistical theory of homogeneous isotropic turbulence, it is sometimes of interest to evaluate the mean square value of certain integrals, such as the pressure fluctuation over a sphere. The purpose of the present note is to give such an evaluation for integrals over a sphere and for similar integrals over spaces of other dimensions. The analysis shows that the final answer can be interpreted in terms of dimensional arguments; provided the length scale used is the *geometrical mean* of the scale of turbulence and the linear scale of the region over which the integral is taken. The results could be applied to the problem of the noise generated by turbulence.

2. Consider, for definiteness, the pressure fluctuation over the surface of a sphere. Extension to the study of other quantities can be easily made. Let the integral be denoted by

$$I = \int p \, dS, \tag{1}$$

where p is the pressure fluctuation at a point P, and the surface integral is extended over a sphere of radius a. We may also write

$$I = \int p' \, dS', \tag{2}$$

*Received April 20, 1953.

where p' is the pressure fluctuation at a point P', and the integral is extended over the same surface of the sphere. Thus, if we calculate I^2 by multiplying (1) and (2), we obtain

$$\langle I^2 \rangle = \int dS \int \langle pp' \rangle \, dS', \tag{3}$$

where $\langle\ \rangle$ enclosing a quantity denotes a statistical average.

In isotropic turbulence, the statistical correlation $\langle pp' \rangle$ depends only on the relative position of the two points P and P'. Thus, the integral

$$J = \int \langle pp' \rangle \, dS' \tag{4}$$

is independent of the position of the point P, and we have

$$\langle I^2 \rangle = 4\pi a^2 J. \tag{5}$$

To evaluate the integral J, we may take the point P as the origin of a system of spherical coordinates. The element of area can be represented very simply if we consider zonal surfaces at a distance r from the origin. In fact,

$$dS = 2\pi r \, dr. \tag{6}$$

Thus,

$$J = 2\pi \langle p^2 \rangle \int_0^{2a} \mathcal{P}(r) r \, dr, \tag{7}$$

where $\langle p^2 \rangle$ is the mean square value of the pressure fluctuation, and $\mathcal{P}(r)$ is the correlation coefficient for pressure.

For very small spheres, we have the approximation

$$\mathcal{P}(r) \sim 1 \qquad \text{for} \qquad 0 \le r \le 2a. \tag{8}$$

Then J may be approximated by

$$J_0 = 4\pi a^2 \langle p^2 \rangle \tag{9}$$

and $\langle I^2 \rangle$ may be approximated by

$$\langle I^2 \rangle_0 = (4\pi a^2)^2 \langle p^2 \rangle \tag{10}$$

This also follows, if we write

$$I = 4\pi a^2 p \tag{11}$$

for very small spheres.

For large spheres, J may be approximated by

$$J_\infty = 2\pi \langle p^2 \rangle \int_0^\infty \mathcal{P}(r) r \, dr. \tag{12}$$

This may be written as

$$J_\infty = 4\pi \langle p^2 \rangle l^2, \tag{13}$$

where l is a scale of turbulence defined by

$$2l^2 = \int_0^\infty \mathcal{P}(r) r \, dr. \tag{14}$$

Then

$$\langle I^2 \rangle = (4\pi a^2)(4\pi l^2)\langle p^2 \rangle. \tag{15}$$

The ratio $\langle I^2 \rangle / \langle I^2 \rangle_0$ is equal to

$$\frac{\langle I^2 \rangle}{\langle I^2 \rangle_0} = \left(\frac{l}{a}\right)^2 \tag{16}$$

In the limit $a \to \infty$, this approaches zero. This is to be expected since $I/4\pi a^2$, being the average value of I over the sphere, approaches zero when the sphere increases indefinitely in size.

In many cases, we are interested in averages over spheres of a size larger but comparable to the scale of turbulence; then (15) and (16) may be used as suitable approximations.

3. From the above arguments, it is obvious that the order of magnitude of $\langle I^2 \rangle$ is independent of the detailed shape of the surface under consideration. Thus, we may write

$$\langle I^2 \rangle = k_2 \langle p^2 \rangle l^2 L^2, \tag{17}$$

where k_2 is a constant, depending somewhat on the shape, l is a scale of turbulence, and L is a typical linear scale of the surface.

The formula (17) may be easily generalized to other dimensions. In general, we have

$$\langle I_n^2 \rangle = k_n \langle p^2 \rangle (lL)^n, \tag{18}$$

for an integral

$$I_n = \int p \, d\tau_n \tag{19}$$

over an n-dimensional space. Thus, (18) can be obtained from (19) by dimensional arguments provided the volume integration is associated with the nth power of the length scale $(lL)^{\frac{1}{2}}$, i.e., the *geometrical mean* of the scale of turbulence and the linear scale of the region over which the integral is taken.

4. Some care must be exercised in estimating such integrals, if the divergence theorem can be used to convert them to integrals of lower dimensions. Consider, for instance, the integral

$$K = \int \frac{\partial u}{\partial x} \, d\tau \tag{20}$$

over a sphere. If we apply (18) directly to (20), we obtain

$$\langle K^2 \rangle = k_3 \langle (\partial u / \partial x)^2 \rangle (lL)^{3/2},$$

or

$$\langle K^2 \rangle \sim \frac{\langle u^2 \rangle}{\lambda^2} \cdot (lL)^3 \tag{21}$$

where λ is Taylor's micro-scale of turbulence. However, this is actually incorrect. If we write (20) in the form

$$K = \int un_x \, dS, \tag{22}$$

then it becomes evident that

$$\langle K^2 \rangle \sim \langle u^2 \rangle (lL)^2. \tag{23}$$

The ratio between the two estimates (21) and (22) is lL/λ^2. It is clear that the direct application of the volume formula over-estimates the integral because of the small scale introduced by the differentiation process. Such occurrences are frequent in the statistical theory of turbulence. For example, the divergence of the Reynolds stress $\tau_{ij} = -\rho\langle u_i u_j \rangle$ is

$$\frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\{-\rho\langle u_i u_j \rangle\} = -\rho\left\langle u_j \frac{\partial u_i}{\partial x_j}\right\rangle.$$

Based on the left-side, the estimate is

$$\frac{\partial \tau_{ij}}{\partial x_j} \sim \frac{\tau}{l}$$

while the right-side would give the (incorrect) estimate

$$\frac{\partial \tau_{ij}}{\partial x_j} \sim \frac{\tau}{\lambda}.$$

In all such cases, the lower estimates are to be taken.

BOOK REVIEWS

Gasdynamik. By K. Oswatitsch. Springer-Verlag, Wien, 1952. viii + 456 pp. $18.60.

To write a comprehensive treatise on a rapidly expanding field of knowledge is not an attractive task: there is always the possibility that new developments may soon make the treatment appear dated or even incomplete. On the other hand, the lack of such a treatise may seriously impede the recruiting of new scientific talent, because newcomers will find it increasingly difficult to work their way through numerous important papers written in various languages and scattered over a great number of technical periodicals. In producing this comprehensive treatise on the dynamics of compressible fluids, the author has therefore rendered a significant service to all interested in the development of this branch of mechanics of continua.

Chapter I contains the necessary thermodynamic background. Chapters II and III are concerned with steady and unsteady flows in one dimension. The fundamental integral theorems are established in Chapter IV. These integral theorems remain valid in the presence of shocks; the differential equations of motion are readily derived from the integral theorems. Mechanical similarity is discussed, and various vortex theorems are presented. Chapter V illustrates the application of the integral theorems to technical problems. Chapter VI is devoted to the general equations for steady inviscid flow and to exact particular solutions of these equations (Prandtl-Meyer flow, axially symmetric conical flow, transformations of Molenbrock and Chaplygin, linearization of Prandtl and Glauert). Chapter VII is concerned with steady subsonic flows (plane or axially symmetric). In particular, the methods of Krahn, Janzen-Rayleigh, Kármán-Tsien, and Ringleb are discussed. Chapter VIII is devoted to steady supersonic flows in two dimensions (slightly disturbed parallel flow, shocks and their interaction, method of character-

istics). Chapter IX surveys the available results regarding transonic flow. Chapter X is concerned with particular steady and unsteady three-dimensional flows (lifting surfaces, conical flows, delta-wing, decelerated wedge). The influence of viscosity is briefly discussed in Chapter XI, and a survey of experimental techniques is given in Chapter XII.

W. PRAGER

An introduction to linear programming. By A. Charnes, W. W. Cooper, and A. Henderson. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London. ix + 74 pp. $2.50.

The first part of the volume contains an economic introduction to the theory of linear programming. The fundamental concepts are presented in connection with a specific problem ("nut mix problem"), and the mathematical argument is kept as simple as possible. The second part is devoted to the mathematical theory of linear programming.

While all concepts with which the non-mathematical reader is not likely to be familiar are fully explained, the second part, nevertheless, requires greater mathematical experience from the reader than the first part. Obviously, there is some duplication of the mathematical work in the two parts, but this is entirely justified in a work which is aimed at such a diversified group of readers.

W. PRAGER

A selection of tables for use in calculations of compressible airflow. Prepared on behalf of the Aeronautical Research Council by the Compressible Flow Tables Panel. The Clarendon Press, Oxford, 1952. viii + 143 pp. $8.00.

The panel consisted of L. Rosenhead (Chairman), W. G. Bickley, C. W. Jones, L. F. Nicholson, C. K. Thornhill and R. C. Tomlinson.

This book of tables is the first of two volumes. A complementary volume consisting of graphs will be published in the near future. Among other tables, there is a series of tables labeled isentropic flow tables, applying to the steady isentropic flow of air; a series of tables, labeled characteristic tables, applying to steady isentropic supersonic flow of air in two dimensions or in three dimensions with axial symmetry; and a series of tables, labeled shock tables, applying to normal and oblique shocks in air.

P. CHIARULLI

Advanced mechanics of materials. By F. B. Seely and J. O. Smith. John Wiley and Sons, Inc., New York, and Chapman and Hall, Ltd., London, 1952. Second edition. xvii + 680 pp. $8.50.

This second edition of the book first published in 1932, and then by Seely only, is essentially a new book. It is written for advanced undergraduate and first year graduate students in engineering. Special emphasis is laid on the engineering evaluation of analyses of problems. Results are sometimes quoted from cited references, e.g., in the discussion of thin plates, and thereby the mathematics is kept at an elementary level. The discussion of energy methods and of plastic instability of struts is commendably clear. Problems and references are provided.

The book is in six parts [Preliminary considerations. Special topics on the strength and stiffness of members subjected to static loads. Localized stress—stress concentration. Energy methods. Influence of small inelastic strains on the load-carrying capacity of members. Introduction to instability—buckling loads.] with three Appendixes [A brief introduction to the mathematical theory of elasticity. The elastic membrane (soap film) analogy for torsion. Properties of an area.]

H. G. HOPKINS

Theory of elasticity and plasticity. By H. M. Westergaard. Harvard University Press, Cambridge Mass., and John Wiley & Sons, Inc., New York, 1952. xii + 176 pp. $5.00.

Professor Westergaard's original plan was to write a definitive work on two and three-dimensional elasticity and to include plasticity. A chapter on history covers analytical and experimental developments in elasticity, photoelasticity, plasticity, analogies, structures, and mechanics of materials. Realizing he would not have time to finish, Dr. Westergaard made a great effort to complete the part which has been published posthumously. An interesting feature is the emphasis placed upon the displacements, strains, and the strain functions from which they may be derived rather than upon stress. The approach throughout employs vector notations and descriptions wherever possible. Galerkin vector solutions of the problems of Kelvin, Boussinesq, Cerruti, and Mindlin are discussed in detail as is the author's own method of the twinned gradient.

D. C. Drucker

Mechanics of vibration. By H. M. Hansen and P. F. Chenea. John Wiley & Sons, Inc., New York and Chapman & Hall, Ltd., London, 1952, xiii + 417 pp. $8.00.

This book is intended as a textbook for both undergraduate and graduate engineering students. In the authors' words, it is to be a "... textbook that covers the all-important basic principles in a thorough fashion and yet is suitable for a student who has had nothing more than an elementary course in dynamics and the standard instruction in mathematics offered to engineering students today". This reviewer is of the opinion that the authors have accomplished their purpose in large measure insofar as undergraduate students are concerned, but that the book is not well suited as a text for graduate students.

Part I treats at great length a) free vibrations without damping, b) harmonic forced vibrations without damping, c) the steady state solution of harmonic forced vibrations with damping, of linear systems having one degree of freedom.

Part II treats systems of several degrees of freedom. In chapter five there is a brief discussion of Lagrange's equations and of the solution of the equations of small oscillations. In the reviewer's opinion this is the weakest chapter in the book because the all-important subject of the fundamental theory of small oscillations is discussed only very sketchily. Evidently the authors were prevented by space limitations from giving a fuller account of this theory. This reviewer believes that a condensation or omission of other topics, for the sake of a more thorough presentation of this theory, would have been desirable, particularly since the book is intended to be suitable for graduate as well as undergraduate students. This chapter contains a few inaccuracies.

Chapter six is the most valuable portion of the book. It contains an exhaustive discussion of the mobility method for the analysis of certain types of systems of several degrees of freedom. The treatment of various lever systems, branched systems and multimass systems will be very useful to anyone faced with the investigation of problems to which this method applies. Since the method is based on the concept of impedance which is most conveniently developed in connection with electrical systems, the reviewer believes that it is unfortunate that no electrical systems are mentioned.

Part II closes with a discussion of two methods of approximate calculation of the frequencies, viz., Holzer's and Graeffe's method.

Part III, entitled "Special Topics", consists of three chapters, one on continuous systems, one on vibrations of transient character, and one on non-linear vibrations. These are well written, but, being relatively short, barely touch on the subjects involved.

There is a large and remarkable collection of useful problems with answers.

The book as a whole is clearly written. The mathematical tools used are all elementary. This reviewer was surprised at the almost complete lack of literature references.

G. W. Morgan

Elasticity in engineering. By Ernest E. Sechler. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1952. ix + 419 pp. $8.50.

This book is intended to bridge the gap between texts on strength of materials and engineering treatises on the theory of elasticity. It consists of three parts: Fundamental Equations and Analysis Methods, Engineering Problems in Stable Structures, Engineering Problems in Instability. The subject matter was evidently selected with a view toward the interest of structural engineers, in general, and aeronautical engineers, in particular.

The emphasis on theory and understanding is rather light. A few examples may suffice to illustrate the point. The local transformation theory for a state of stress is treated on the assumption of a homogeneous stress field and under "neglect" of the body forces. In a chapter on stress functions Maxwell's stress functions are unearthed and applied exclusively to the representation of a uniform stress field. Under the heading "Uniqueness of Energy Solution" the author claims to prove that for an elastic body in equilibrium under a single concentrated force the displacement at the point of application of the force is unique. This reviewer is unable to follow the argument.

E. Sternberg

Mechanics. By A. Sommerfeld. Translated from the German by M. O. Stern. Academic Press, Inc., New York, 1952. xiv + 289 pp. $6.50.

This book constitutes volume 1 of the author's *Lectures on theoretical physics.* It is concerned with the mechanics of systems of a finite number of degrees of freedom. (Systems with an infinite number of degrees of freedom are treated in vol. 2 entitled *Mechanics of deformable bodies* an English translation of which was published in 1950.) The following chapter headings will give an idea of the scope of the book: Mechanics of a particle—Mechanics of systems, principle of virtual work, and d'Alembert's principle—Oscillation problems—The rigid body—Relative motion—Integral variational principles of mechanics and Lagrange's equations for generalized coordinates—Differential variational principles of mechanics—The theory of Hamilton—Problems. There is no need to comment here on the merits of Sommerfeld's exposition; the translation is well done.

W. Prager

Grenzschicht-Theorie. By H. Schlichting. G. Braun, Karlsruhe, 1951. xv + 481 pp. $10.00.

The key papers on boundary layer theory are scattered over many technical journals some of which are not readily accessible. Moreover, development has been very intense in this field for the last three decades and is still continuing at a rapid pace. These circumstances make a comprehensive treatment such as Professor Schlichting's book particularly valuable, for the research worker as well as the newcomer to the field. The subject matter is organized into four parts: Basic laws of viscous flow—Laminar boundary layers—Transition from laminar to turbulent flow—Turbulent boundary layers. In addition to the material indicated by the heading, the first part contains a valuable review of exact solutions of the Navier-Stokes equations. The second part presents the theory of the laminar boundary layer for plane steady flow, for axially symmetric, and three-dimensional flow, and for unsteady, in particular oscillatory flow. Temperature boundary layers and boundary layers in compressible flow are also discussed. The third part is rather brief, covering only thirty-one pages. The discussion of turbulent flows in the fourth part is based on the older hypotheses regarding mixing length or similarity rather than the more recent statistical concepts. The exposition is clear throughout and easy to follow.

W. Prager

Advanced dynamics. By E. Howard Smart. Macmillan and Co., Ltd., London, 1951. Vol. I, xi + 419 pp. Vol. II, xi + 420 pp. $12.00 per set.

One outstanding feature of this two-volume work is its tremendous scope. No previous knowledge of mechanics is assumed, and the subject of the dynamics of a particle and of rigid bodies is developed slowly, methodically, and in great detail from the very beginning through to the more advanced topics such as the Hamilton theory. Many topics, which in many books on mechanics are given only a cursory treatment, appear here in great detail. The text abounds in examples worked out at great length. There are extensive exercises for the reader, many of the problems being quite interesting. As for prerequisites, a first course in differential equations should, for the most part, suffice for volume I and the first half of volume II. However, for the last half of volume II considerable more mathematical maturity seems advisable.

Volume I deals with the dynamics of a particle, almost all of the material being on two-dimensional topics. The kinematics of a particle is first introduced, then Newton's laws of motion. There follow extensive treatments of the rectilinear motion of a particle, as well as the plane motion of a particle investigated by the use of acceleration components in the directions of rectangular cartesian coordinates and plane polar coordinates. Included here are broad treatments of the ballistic problem and central orbits. Among other topics are the motion of a particle on a curve or surface and the motion of chains. Volume I concludes with a chapter on the motion of a particle in three dimensions.

Volume II deals with the dynamics of a rigid body, and with certain advanced topics. Two-dimensional motions are first considered at some length. Included here is an extensive treatment of impulses. There is one chapter on three-dimensional motions. About the last half of volume II is occupied with more advanced topics including the top and gyroscope, Lagrange's equations of motion, the Hamilton theory, and the theory of small oscillations of conservative systems.

The reviewer has but two criticisms to make, both of which may be matters of opinion. First, the title of the work seems a bit misleading, since the subject of dynamics is developed in this work right from the beginning, and only the last half of volume II deals with what is commonly called "advanced dynamics". Secondly, almost no use of vectors is made. The notion of a vector is introduced after it has been thoroughly motivated by the velocity and acceleration of a particle. But the only operation introduced from the algebra and calculus of vectors is addition. Very many derivations and equations are thus expressed in terms of components, which is relatively awkward and space consuming. Also, no special notation is used for vectors, so that it is sometimes difficult to ascertain whether vector or scalar character is implied.

G. E. Hay

Numerical solution of differential equations. By W. E. Milne. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1953. xi + 275 pp. $6.50.

As is stated in the preface, the book attempts to acquaint the reader with the principal techniques for the numerical solution of ordinary and partial differential equations.

Chapters 1 (Introduction), 3 (Analytical Foundations), and 9 (Linear Equations and Matrices) contain useful background material not directly connected with numerical techniques of integrating differential equations. Elementary methods of solving ordinary differential equations are discussed in Chapter 2 (point-slope formula, trapezoidal formula). Chapter 4 is devoted to methods based on numerical integration and Chapter 5 to the method of Runge-Kutta and methods involving higher derivatives. Systems of ordinary differential equations are treated in Chapter 6, and two-point boundary value problems in Chapter 7. Chapter 8 deals with explicit methods for parabolic and hyperbolic partial differential equations, and Chapter 10 with implicit methods for elliptic equations. Numerical methods of obtaining characteristic numbers are presented in Chapter 11. Appendices are concerned with round-off errors, large-scale computing machines, and the Monte-Carlo method.

The exposition is careful and clear. The detailed numerical examples will enable the practical computer to acquaint himself with the characteristic features of the various methods. Reflecting the considerable practical experience of the author, the book is a most welcome addition to the literature on this important field.

W. Prager

Vorlesungen über die Theorie der Integralgleichungen. By I. G. Petrovskij. Translated from the Russian by R. Herschel. Physica-Verlag, Würzburg, 1953. 100 pp. $7.80.

The little volume contains an excellent exposition of the Fredholm-Hilbert-Schmidt theory of linear integral equations.

W. PRAGER

Thermionic vacuum tubes and their applications. By W. H. Aldous and Sir Edward Appleton. John Wiley & Sons, Inc., and Methuen & Co. Ltd., London, 1952. vii + 160 pp. $2.00.

This monograph treats the basic theory and applications of high vacuum tubes including velocity modulated tubes (klystrons, etc.). Written in a clear and concise manner it serves the authors' purpose as an introduction to electronic circuits while the chapter on limits to amplification is worth reading for the more advanced worker.

S. L. LEVY

International Congress of Mathematicians 1954

The International Congress of Mathematicians 1954 will be held in Amsterdam from September 2nd to September 9th under the auspices of „Het Wiskundig Genootschap" (The Mathematical Society of the Netherlands). It is the sincere hope of the „Wiskundig Genootschap" that the Congress 1954, which will be open to all mathematicians from all parts of the world, will be a fertile international gathering.

The Organizing Committee has invited a number of outstanding mathematicians to deliver one-hour addresses, hoping that in this way a survey of the recent development in the whole field of mathematics may be furnished.

There will be seven sections, viz:

1 Algebra and Theory of Numbers.
2 Analysis.
3 Geometry and Topology.
4 Probability and Statistics.
5 Mathematical Physics and Applied Mathematics.
6 Logic and Foundations.
7 Philosophy, History, and Education.

In each of these sections half-hour addresses will be delivered by experts on invitation of the Organizing Committee. Moreover short lectures will be given by members of the Congress who have applied beforehand to the Organizing Committee. The time allotted for each short lecture will be 15 minutes. It will depend on the number of these short lectures whether and how the sections will be divided into subsections.

The Organizing Committee is planning several entertainments and also a number of interesting excursions.

There will be two categories of membership in the Congress:

regular members (*members*) who will be entitled to participate in the scientific and social activities of the Congress and to receive the Proceedings of the Congress, and
associate members who, accompanying regular members of the Congress, do not participate in the scientific programme nor receive the Proceedings, but will be entitled to many of the privileges of membership.

The fees to be paid by members and by associate members have not yet been definitely fixed but presumably they will not exceed the amount of fifty guilders (about $14.—) for members and twenty guilders ($5.50) for associate members.

Those who wish to attend the Congress are requested to communicate their name (with degrees, qualifications, etc.) and full address to the Secretariat as soon as possible. They will receive a more detailed communication which will be sent out in the course of 1953.

Amsterdam, 2d Boerhaavestraat 49 *The Organizing Committee*

Contents (*Continued from Back Cover*)

SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ('), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set above letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which follow the letter should be used.

Square roots should be written with the exponent ½ rather than with the sign √.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

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New Book Announcements

HIGHER TRANSCENDENTAL FUNCTIONS, Volume I
[illegible] California Institute of Technology
[illegible]
This is the first of a three volume work of reference which will provide a comprehensive account of [illegible] functions. [illegible] hypergeometric functions, and generalized hypergeometric functions.

HIGHER TRANSCENDENTAL FUNCTIONS, Volume II
[illegible] Project Staff, California Institute of Technology
Edited by A. [illegible]
In press
This is the second volume of a three-volume set [illegible] with the California [illegible] functions, parabolic cylinder functions, error functions, exponential, sine, cosine integral and related functions, orthogonal polynomials, elliptic functions and integrals.

STABILITY THEORY OF DIFFERENTIAL EQUATIONS
By Richard Bellman, The Rand Corporation. *International Series in Pure and Applied Mathematics*. [illegible]
Original work and research [illegible] an important aspect of advanced mathematics [illegible] to the modern theory of stability and asymptotic behavior of solutions of linear and nonlinear differential equations.

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